NUI MAYNOOTH

Ollscoil na hÉireann Má Nuad

OLLSCOIL NA hÉIREANN, MÁ NUAD

THE NATIONAL UNIVERSITY OF IRELAND MAYNOOTH

First Computer Science and Software Engineering Examination

AUTUMN
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SE120
DISCRETE STRUCTURES 2

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Time allowed: 2 hours

Answer ***three*** questions

All questions carry equal marks

1. Consider the following Hoare triple.
   $\{x > 0\}$
   $a := 1;\ b := -1;$
   while $(a \leq x)$ **do**
   $\quad b := b + 1;\ a := a * 2;$
   od
   $a := a/2;$
   $\{2^b \leq x \wedge 2^{b+1} > x \wedge a = 2^b\}$

   Appropriately reformatting the triple gives the following, where P is the loop invariant and Q is a predicate that must be calculated in order to prove the triple is correct.
   $\{x > 0\}$
   $a := 1;\ b := -1;$
   $\{P\}$
   while $(a \leq x)$ **do**
   $\quad b := b + 1;\ a := a * 2;$
   od
   $\{P \wedge \neg(a \leq x)\}$
   $\{Q\}$
   $a := a/2;$
   $\{2^b \leq x \wedge 2^{b+1} > x \wedge a = 2^b\}$

   Assume that all variables are integer variables. Complete each of the following steps.

   (a) Show that $Q = \{2^b \leq x \wedge 2^{b+1} > x \wedge a = 2^{b+1}\}$ by a calculation using the assignment axiom. [5 marks]

   (b) Show that $P = \{a/2 \leq x \wedge a = 2^{b+1}\}$ using the fact that $P \wedge \neg(a \leq x) \rightarrow Q$. [7 marks]

   (c) Prove the loop is correct by proving $\{P \wedge (a \leq x)\}\ b := b + 1; a := a * 2\ \{P\}$. [10 marks]

   (d) Explain what remains to be done in order to prove that the Hoare triple is correct. [3 marks]

2. (a) For each of the following predicates, state one predicate that is strictly stronger, and one that is strictly weaker. In each case, identify which is which. Each predicate must be unique: do not re-use an answer to one part in another part. [12 marks]

   i. $(x = 0)$
   ii. $(x > 0 \wedge y = 0)$
   iii. $(x = 0 \wedge y = 0)$

   (b) For each of the following sets, prove that it is countable or prove that it is uncountable. Assume that $\mathbb{Z}^-_{\text{odd}} = \{x : x \in \mathbb{Z}, x < 0 \wedge \text{mod}(\text{abs}(x), 2) = 1\}$ and assume that $\mathbb{R}^1_0 = \{x : x \in \mathbb{R}, 0 \leq x \leq 1\}$. [13 marks]

   i. $\mathbb{N} \times \mathbb{Z}$
   ii. $\mathbb{N} \cup \mathbb{Z}^-_{\text{odd}}$
   iii. $\mathbb{R}^1_0$

3. (a) Write out three elements of each of the following relations. [9 marks]

   i. SQRT = $\{(a, b) : a \in \mathbb{N}, b \in \mathbb{R}, \sqrt{a} = b\}$

   ii. SQR = $\{(a, b) : a \in \mathbb{Z}, b \in \mathbb{N}, a^2 = b\}$

   iii. SWAP = $\{((a, b), (c, d)) : a, b, c, d \in \mathbb{N}, a = d, b = c\}$

   (b) For each of the relations in part (a) of this question, state whether it is a function or not. For each relation that is not a function, redefine it without changing its meaning so that it is a function. [6 marks]

   (c) Let $\sim$ be a relation on $\mathbb{Z}$ defined by $x \sim y$ iff $(x > 5) = (y > 5)$. Use this relation to partition $\mathbb{Z}$. [5 marks]

   (d) Given the binary relation $R = \{(0, 2), (1, 2), (2, 3), (3, 4)\}$ construct each of the following compositions. [5 marks]

   i. R^2

   ii. R^3

   iii. R^4

4. (a) Each of the following Hoare triples claims to correctly perform the swapping process. The first one uses a temporary variable. The second does not. Prove that each triple is correct. Assume that all variables are integer variables. [15 marks]

   i. $\{x < y\}$ temp $:= x; x := y; y :=$ temp $\{y < x\}$

   ii. $\{x < y\}$ $y := y + x; x := y - x; y := y - x$ $\{y < x\}$

   (b) Let the language $L = \{a, ab, aab, aba, abab, baaa\}$. Define an equivalence relation of your choice that partitions L into the following numbers of classes. Write out the contents of the classes in each case. [10 marks]

   i. An equivalance relation that partitions L into exactly two classes

   ii. An equivalance relation that partitions L into exactly three classes

Rules that can be applied in any question

Implication truth table:

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

Simplification (Simp): $$\frac{A \wedge B}{A}$$

Addition (Add): $$\frac{A}{A \vee B}$$

Conjunction (Conj): $\dfrac{A, B}{A \wedge B}$

Transitive: $\dfrac{a > b \wedge b > c}{a > c}$

Conditional Proof Rule (CP): If there is a proof of B from the assumption that A is true (i.e. if B can be derived from A), then $A \rightarrow B$

Assignment Axiom (AA): $\{Q(x/t)\}\, x := t\, \{Q\}$

Consequence Rule: $\dfrac{P \rightarrow R \quad \text{and} \quad \{R\}\, S\, \{Q\}}{\{P\}\, S\, \{Q\}}$

Composition Rule: $\dfrac{\{P\}\, S_1\, \{R\} \quad \text{and} \quad \{R\}\, S_2\, \{Q\}}{\{P\}\, S_1; S_2\, \{Q\}}$

If-Then Rule: $\dfrac{\{P \wedge C\}\, S\, \{Q\} \quad \text{and} \quad P \wedge \neg C \rightarrow Q}{\{P\}\, \textbf{if}\, C\, \textbf{then}\, S\, \{Q\}}$

If-Then-Else Rule: $\dfrac{\{P \wedge C\}\, S_1\, \{Q\} \quad \text{and} \quad \{P \wedge \neg C\}\, S_2\, \{Q\}}{\{P\}\, \textbf{if}\, C\, \textbf{then}\, S_1\, \textbf{else}\, S_2\, \{Q\}}$

While Rule: $\dfrac{\{P \wedge C\}\, S\, \{P\}}{\{P\}\, \textbf{while}\, C\, \textbf{do}\, S\, \{P \wedge \neg C\}}$

Statements that can be quoted without proof:

1. $\mathbb{N}$ is countable
2. Any set that has a bijection with a subset of $\mathbb{N}$ is countable
3. Let $B = A_1 \cup A_2 \cup \ldots \cup A_n$. If each A_i is countable then B is countable. If at least one A_i is uncountable then B is uncountable.
4. Let $B = A_1 \times A_2 \times \ldots \times A_n$. If each A_i is countable then B is countable. If at least one A_i is uncountable then B is uncountable.