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NATIONAL UNIVERSITY OF IRELAND, MAYNOOTH

FIRST COMPUTER SCIENCE AND SOFTWARE ENGINEERING EXAMINATION

SAMPLE SOLUTIONS 2003

PAPER SE120

DISCRETE STRUCTURES II

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1. (a) As a consequence of $\{P\}S\{R\}$ evaluating to true, we know that if predicate P evaluates to true, and S is executed, then R will be true after S terminates. [3 marks]

(b) From the If-Then-Else Rule we have to prove (i) $\{P \wedge C\}\, S_1\, \{Q\}$ and (ii) $\{P \wedge \neg C\}\, S_2\, \{Q\}$. [12 marks]

(i) Prove $\{\text{true} \wedge \text{odd}(x)\}\; y := x + 2\; \{\text{odd}(y) \wedge y > x\}$

1. $\{\text{odd}(x+2) \wedge x+2 > x\}\; y := x+2\; \{\text{odd}(y) \wedge y > x\}$	AA
2. true $\wedge$ odd(x)	precondition
3. odd(x)	2,AND-simplification
4. odd$(x+2)$	3,T
5. $x+2 > x$	T
6. odd$(x+2) \wedge x+2 > x$	4,5
7. true $\wedge$ odd$(x) \rightarrow$ odd$(x+2) \wedge x+2 > x$	2,6
8. $\{\text{true} \wedge \text{odd}(x)\}\; y := x+2\; \{\text{odd}(y) \wedge y > x\}$	1,7,consequence

(ii) Prove $\{\text{true} \wedge \neg\text{odd}(x)\}\; y := x + 1\; \{\text{odd}(y) \wedge y > x\}$. Although not essential, let us first simplify this to $\{\text{even}(x)\}\; y := x+1\; \{\text{odd}(y) \wedge y > x\}$

9. $\{\text{odd}(x+1) \wedge x+1 > x\}\; y := x+1\; \{\text{odd}(y) \wedge y > x\}$	AA
10. even(x)	precondition
11. odd$(x+1)$	10,T
12. $x+1 > x$	T
13. odd$(x+1) \wedge x+1 > x$	11,12
14. even$(x) \rightarrow$ odd$(x+1) \wedge x+1 > x$	10,13
15. $\{\text{even}(x)\}\; y := x+1\; \{\text{odd}(y) \wedge y > x\}$	9,14,consequence
QED	8,15,If-Then-Else Rule

(c) There are four equivalence classes [10 marks]

i. $[0] = \{0, 4, 8, \ldots\}$
ii. $[1] = \{1, 5, 9, \ldots\}$
iii. $[2] = \{2, 6, 10, \ldots\}$
iv. $[3] = \{3, 7, 11, \ldots\}$

and indeed these do form a partition of $\mathbb{N}$, because $[0] \cup [1] \cup [2] \cup [3] = \mathbb{N}$ and $[0] \cap [1] \cap [2] \cap [3] = \emptyset$. Then, one possible expression for these equivalence classes (and one possible answer to this question) is $[n] = \{x | x \in \mathbb{N}, x \bmod 3 = n\}$.

2. (a) The language consists of all pairs (a, b) where a is a list of integers, and b is list sorted in ascending order containing the exact same integers as in a. For example, the pair $((7, 55, 3, 23), (3, 7, 23, 55))$ would be in this language. If we can accept this language, then we can sort lists of integers. [3 marks]

(b) $R^3 = \{(1, 4), (2, 5)\}$ [3 marks]

(c) /* This proof would involve equating the set of all problems with the set of all languages over some alphabet (the alphabet $\{0, 1\}$, for example) and then a diagonalisation proof showing that this set cannot be ordered. */ [19 marks]

3. (a) Language acceptance (language recognition) problems are of interest to computer scientists because it is the most general form of computation. Every computation that a particular machine (finite automaton, pushdown automaton, Turing machine) can perform can be represented as a language that the machine can accept. Languages that the machine cannot accept correspond exactly to computations that that machine cannot perform. [5 marks]

(b) In order to calculate the loop invariant P, we must first reformat the while loop according to the While Rule. This gives us [10 marks]

$\{x > 0 \wedge z \geq x\}$
$y := 0;$
$\{P\}$
while $(x + y) < z$ **do**
  $y := y + 1$
od
$\{P \wedge \neg(x + y < z)\}$
$\{x + y = z\}$

From this, we can see that the second last line must imply the last line:
$P \wedge \neg(x + y < z) \rightarrow x + y = z$
and this gives us a method of solving for P.

The expression can be simplified to
$P \wedge x + y \geq z \rightarrow x + y = z$

The weakest predicate P (the predicate for P that allows the most possibilities) is therefore $P = x + y \leq z$.

(c) i. Uncountable [1 marks]
ii. Countably infinite [1 marks]
iii. Countably infinite [1 marks]
iv. Finite [1 marks]

(d) A relation is an equivalence relation if it is reflexive, symmetric, and transitive. So, to turn R into an equivalence relation we need only find the union of the reflexive, symmetric, and transitive closures of R. [6 marks]
$r(R) = \{(1,1),(2,2),(2,3),(3,2),(3,3),(4,4),(4,5),(5,5)\}$
$s(R) = \{(1,1),(2,2),(2,3),(3,2),(4,5),(5,4)\}$
$t(R) = \{(1,1),(2,2),(2,3),(3,2),(3,3),(4,5)\}$
Let $R' = r(R) \cup s(R) \cup t(R)$
$= \{(1,1),(2,2),(2,3),(3,2),(3,3),(4,4),(4,5),(5,4),(5,5)\}$. R' is an equivalence relation.

4. (a) The set X is countable because a bijection exists between the elements of X and $\mathbb{N}$. One such function that maps naturals to X is $f : \mathbb{N} \rightarrow X$, $f(a) = a \div 100$. This function is a bijection because its inverse ($f^{-1} : X \rightarrow \mathbb{N}$, $f(a) = a \times 100$) maps X back to $\mathbb{N}$. [8 marks]

(b) i. To prove the truth or falsity of $\{x > 0\}\, x := x * x; x := x \div 2\, \{x^4 = 10\}$ we apply the composition rule and the assignment axiom. [6 marks]

1. $\{(x \div 2)^4 = 10\}\, x := x \div 2\, \{x^4 = 10\}$	AA
2. $\{(x * x \div 2)^4 = 10\}\, x := x * x\, \{(x \div 2)^4 = 10\}$	1,AA
3. $x > 0$	precondition
4. $x > 0 \not\rightarrow (x * x \div 2)^4 = 10$	

We can prove that the left hand side does not imply the right hand side in line 4, by identifying a single value for x which satisfies the left hand side but which does not satisfy the right hand side. For example, $x = 1$. Therefore, this program is not correct (with respect to its precondition and postcondition).

ii. To prove the truth or falsity of {true} $x := 5; x := x + 1\, \{x > 5\}$ we apply the composition rule and the assignment axiom. [6 marks]

1. $\{x + 1 > 5\}\, x := x + 1\, \{x > 5\}$	AA
2. $\{5 + 1 > 5\}\, x := 5\, \{x + 1 > 5\}$	1,AA
3. true	precondition
4. $5 + 1 > 5$	T
5. true $\rightarrow 5 + 1 > 5$	T$\rightarrow$T from truth table
6. {true} $x := 5\, \{x + 1 > 5\}$	2,5,consequence
QED	1,6,composition

(c) The relation $\sim$ is an equivalence relation so the partition of $\mathbb{N}$ can be obtained by writing out the equivalence classes: [5 marks]
$[0] = \{0, 10, 20, \ldots\}$ $[1] = \{1, 11, 21, \ldots\}$ $[2] = \{2, 12, 22, \ldots\}$
$[3] = \{3, 13, 23, \ldots\}$ $[4] = \{4, 14, 24, \ldots\}$ $[5] = \{5, 15, 25, \ldots\}$
$[6] = \{6, 16, 26, \ldots\}$ $[7] = \{7, 17, 27, \ldots\}$ $[8] = \{8, 18, 28, \ldots\}$
$[9] = \{9, 19, 29, \ldots\}$