Ollscoil Mhá Nuad
Maynooth University

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AUTUMN
2024-2025

CS605
The Mathematics and Theory of Computer Science

Asst. Prof. G. Di Liberto, Assoc. Prof. A. Mooney, Prof. T. Naughton

Time allowed: 3 hours

Answer all ***seven*** questions

All questions carry equal marks

Instructions

	Yes	No	N/A
Formulae and Tables book allowed *(i.e. available on request)*		X	
Formulae and Tables book required *(i.e. distributed prior to exam commencing)*		X	
Statistics Tables and Formulae allowed *(i.e. available on request)*		X	
Statistics Tables and Formulae required *(i.e. distributed prior to exam commencing)*		X	
Dictionary allowed *(supplied by the student)*		X	
Non-programmable calculator allowed		X	
Students required to write in and return the exam question paper	X		
One physical (paper) copy of textbook Michael Sipser, Introduction to the Theory of Computation, without annotations or extra pages (supplied by the student)	X		
Students required to sign the declaration page at the end of this document	X		

Only the answers on this exam question paper will be marked – rough work on extra pages will not be marked. If your answer requires more space than the exam question paper allows, at the bottom of your answer in the exam question paper, specify the page number(s) in your answer booklet where your answer continues.

Write your name, student number, and desk number below.

Name .... Student no................................ Desk no..............

[10 marks]

1 (a) Prove that the following language is not regular.

L1A = $\{w \in \{0, 1\}^*$: $|w|$ is odd, w has a 1 in the middle$\}$ e.g. an example word in the language is 0101011 [4 marks]

(b) Prove that the following language is not context-free.

L1B = $\{w|v_1s_1|v_2s_2...|v_ns_n : w \in \{a, b\}^*, |w| = n, v_i \in \{a, b\}, w = v_1v_2...v_n$, and $s_i \in \{a\}^* \}$ e.g. an example word in the language is abba|aaa|b|ba|aa [6 marks]

You must use this exam question sheet for your proofs.

<u>Proofs</u>

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Page in answer booklet where this answer continues: ______

[10 marks]

2 (a) Prove that the following language is decidable.

L2A is the language of finite automata that accepts all words beginning with symbol a and no other words. It is defined formally as
L2A = {<*M*>: *M* is a finite automaton and $\mathcal{L}(M)$ = {*w* ∈ {a, b}*, *w* begins with a}}. [5 marks]

(b) Prove that the following language is Turing-recognisable.

L2B = {<*J*, *a*, *b*, *c*> : *J* is a Java program, *a, b,* and *c* are string variables declared in *J*, and when *J* is run, both *a* and *b* have values that are substrings of *c* at the same time at least once} [5 marks]

You must use this exam question sheet for your proofs.

<u>Proof</u>

Page in answer booklet where this answer continues: ______

[10 marks]

3 Prove that the problem associated with language L3 defined below is undecidable. You are given that HALT = {<*M*, *w*>: *M* is a Turing machine and *M* halts on *w*} is undecidable. Use the template provided to perform a mapping reduction. You must give your answer on this exam question sheet.

The language L3 is defined as
L3 = {<*M*>: *M* is a Java program that instantiates (constructs) an object of type MyClass at least once during the running of *M*}.

Note, in the template below, some blanks have a small subscript number. Blanks with the same subscript number must have the same value. Blank *N* = "..." is worth 10 marks, and each other blank scores –1 if incorrect.

<u>Proof</u> . . . .

Optional: The complement $\overline{\text{L3}}$ = ...

We will use a mapping reduction to prove the reduction

. . . . ≤ . . . .

Assume that . . . .$_1$ is decidable.

The transition function *f* that maps instances of . . . . to instances of

. . . . is given by TM *F* given by the following pseudocode.

F = "On input <. . . . .$_2$>:

1. Construct the following *N* given by the following pseudocode.

   N = ". . . . .

   . . . .

   . . . .

   . . . .

   . . . .

   . . . ."

2. Output <. . . . .$_3$>."

Now, <. . . . .$_3$> is an element of . . . . iff <. . . . .$_2$> is an element of . . . .

So, using *f* and the assumption that . . . .$_1$ is decidable, we can decide

. . . . A contradiction.

Therefore, ...$_1$ is undecidable. (Also, the complement of . . . .$_1$ is undecidable; the complement of an undecidable language is undecidable.)

[10 marks]

4 In the case of L3 defined earlier, and its complement $\overline{L3}$, prove whether each language is Turing-recognisable or not Turing-recognisable.

You can refer to your previous proof that L3 is undecidable in your answer. Note, you are not told whether L3 is Turing-recognisable or not. You must give your answer on this exam question sheet.

Proof

Page in answer booklet where this answer continues: ______

[10 marks]

5 Prove that the problem associated with language L5 defined below is undecidable. You are given that HALT = {<*M*, *w*>: *M* is a Turing machine and *M* halts on *w*} is undecidable. Use the template provided to perform a mapping reduction. You must give your answer on this exam question sheet.

The language L5 is defined as
L5 = {< *J* >: *J* is a Java program and while running, *J* never prints anything to the screen}.

Note, in the template below, some blanks have a small subscript number. Blanks with the same subscript number must have the same value. Blank *N* = "..." is worth 10 marks, and each other blank scores –1 if incorrect.

<u>Proof</u> . . . .

Optional: The complement $\overline{\text{L5}}$ = ...

We will use a mapping reduction to prove the reduction

. . . . ≤ . . . .

Assume that . . . .$_{1}$ is decidable.

The transition function *f* that maps instances of . . . . to instances of

. . . . is given by TM *F* given by the following pseudocode.

F = "On input <. . . . .$_{2}$>:

1. Construct the following *N* given by the following pseudocode.

   N = ". . . . .

   . . . .

   . . . .

   . . . .

   . . . .

   . . . ."

2. Output <. . . . .$_{3}$>."

Now, <. . . . .$_{3}$> is an element of . . . . iff <. . . . .$_{2}$> is an element of . . . .

So, using *f* and the assumption that . . . .$_{1}$ is decidable, we can decide . . . . A contradiction.

Therefore, ...$_{1}$ is undecidable. (Also, the complement of . . . .$_{1}$ is undecidable; the complement of an undecidable language is undecidable.)

[10 marks]

6 Prove that each of these problems is in $\mathcal{NP}$. You must use this exam question sheet for your proofs.

(a) 3D-MATCHING = {<*A, k*> : *A* is a set of triples $A \subseteq (X \times Y \times Z)$ where *X*,*Y*,*Z* are disjoint finite sets (they share no elements) each containing *k* elements, and there exists a subset $A' \subseteq A$, also of size *k*, with the property that no element appears in multiple triples in *A'* }. [5 marks]

As examples, where *X*={0,1}, *Y*={x, y}, *Z* = {+, -},
<{(1, y, +), (0, x, -), (1, x, +)}, 2> ∈ 3D-MATCHING
<{(1, y, +), (0, y, -), (1, x, -)}, 2> ∉ 3D-MATCHING
<{(1, y, +), (0, y, -), (1, x, -)}, 1> ∈ 3D-MATCHING

(b) SUBSET-SUM-U = {<*S*, *t*> : $S = \{x_1, ..., x_m\}$, where each $x_i \in \{0\}^*$, where t $\in \{0\}^*$, and for some subset $\{y_1, ..., y_n\} \subseteq \{x_1, ..., x_m\}$, it is the case that $\Sigma|y_i| = |t|$ (i.e. the sum of the lengths of strings y_i equals the length of *t*) }. [5 marks]

As examples,
<{0, 00, 000}, 0000> ∈ SUBSET-SUM-U, and
<{00, 000, 00000}, 0000> ∉ SUBSET-SUM-U.

<u>Proofs</u>

Page in answer booklet where this answer continues: ______

[10 marks]

7 The HITTING-SET-0 problem is defined as HITTING-SET-0 = $\{<T, k> : T = \{S_1, ..., S_n\}$ is a system of sets and k is an integer, where each set $S_i \subseteq X$ is a subset of an underlying set X = $\{x_1, ..., x_m\}$, where each $x_i \in \{0\}^*$ is an integer represented in unary notation, and there exists another subset $H \subseteq X$ of size k that has a nonzero intersection with each $S_0, ..., S_n\}$.

Prove that HITTING-SET-0 is $\mathcal{NP}$-complete. You are given that 3-SAT is $\mathcal{NP}$-complete. You must use this exam question sheet for your proof.

<u>Proof</u>

Page in answer booklet where this answer continues: ______

OLLSCOIL NA hÉIREANN MÁ NUAD

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Declaration

To be signed by each student and returned with their exam paper at the end of the examination

i. I have searched through my copy of M. Sipser, *Introduction to the Theory of Computation*, any edition, (the Sipser book) and it does not contain any extra pages or annotations (except for annotations that correct minor typographical errors).

ii. I understand that by failing to notify an invigilator of any annotations or extra pages in my copies of the Sipser book, I will receive a mark of zero in this examination. This does not affect any further disciplinary actions that the University may wish to take.

iii. I understand also that directly copying large amounts of material from the Sipser books without substantially tailoring it to the question asked will not gain any marks for that question.

Name (capital letters) ______________________________

Student number ____________________ Desk number ____________

Signature ______________________ Date ________________