CS403/SE307/CS355 - Computation and Complexity
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Lab 4 - Decidability and undecidability - Group B - 11 Nov 2003 - Sample solutions

Question 1(i)
Proof that $\mathbf{HASVAR_J}$ is Turing recognisable. We construct a TM M to recognise HASVAR_J as follows.

M = "On input $\langle J, v \rangle$:
1. Search through J looking for a declaration for v.
2. Where it is declared, check what value it is initialised to.
3. If it is initialised to zero, accept."

M recognises HASVAR_J. (I.e. M is guaranteed to accept all instances of HASVAR_J and M will never accept pairs $\langle J, v \rangle$ that are not instances of HASVAR_J.) Therefore, HASVAR_J is Turing-recognisable.

Question 1(ii)
Proof that $\mathbf{\overline{HASVAR_J}}$ is Turing recognisable. We construct a TM M to recognise $\overline{\text{HASVAR}_\text{J}}$ as follows.

M = "On input $\langle J, v \rangle$:
1. Search through J looking for a declaration for v.
2. If no declaration is found, accept.
3. Where it is declared, check what value it is initialised to.
4. If it is not initialised to zero, accept."

M recognises $\overline{\text{HASVAR}_\text{J}}$. Therefore, $\overline{\text{HASVAR}_\text{J}}$ is Turing-recognisable.

Question 1(iii)
Yes. (Because it and its complement are Turing recognisable.)

Question 2(i)
Proof that $\mathbf{VARZERO_J}$ is Turing recognisable. We construct a TM M to recognise VARZERO_J as follows.

M = "On input $\langle J, v \rangle$:
1. If v is initialised to zero in J, accept.
2. Run J.
3. After each line of J is executed, check if v has value zero.
4. If it is zero, accept."

M recognises VARZERO_J. Therefore, VARZERO_J is Turing-recognisable.

Question 2(ii)

Proof that VARZERO_J is undecidable. We will use a mapping reduction to prove the reduction $\text{HALTS}_\text{J} \leq \text{VARZERO}_\text{J}$. Assume that VARZERO_J is decidable. The function f that maps instances of HALTS_J to instances of VARZERO_J is performed by TM F given by the following pseudocode.

F = "On input $\langle J \rangle$:

1. Construct the following M' given by the following pseudocode.

   M' = "class Mprime {
   public static void main(String args[]) {
   int $v = -1$;
   $J()$;
   $v{+}{+}$;
   }
   }"
2. Output $\langle M', v \rangle$"

Now, $\langle M', v \rangle$ is an element of VARZERO_J iff $\langle J \rangle$ is an element of HALTS_J. So using f and the assumption that VARZERO_J is decidable, we can decide HALTS_J. A contradiction. Therefore, VARZERO_J is undecidable.

Question 2(iii)

No. (Because VARZERO_J is undecidable and Turing recognisable, its complement cannot also be Turing recognisable.)