CS403/SE307/CS355 - Computation and Complexity Department of Computer Science National University of Ireland, Maynooth

T Naughton, NUIM http://www.cs.may.ie/~tnaughton/teaching/se307

Lab 3 - Countable and uncountable sets - Group A - 21 Oct 2003 - Sample solutions

Question 1(i)

Proof that the union of k **countable sets is a countable set.** We will prove that the union is countable by enumerating it. This is straightforward because each of the k individual sets can be enumerated. The following TM M performs the enumeration.

M = "On input $\langle \rangle$:

- 1. Set some counter i equal to 0.
- 2. Print out the i^{th} element of each of the k sets.
- 3. Increment i and repeat."

If you pick any particular element from the union of the k sets, and run M, then M will eventually print out that element. Therefore the union is countable.

Question 1(ii)

The illustration below shows how that union (where k = 3) could be enumerated.



Question 1(iii)

Yes, the union of a countable number of countable sets is itself a countable set. We could enumerate the union of a countably infinite amount of countably infinite sets $A_0, A_1, A_2, A_3, \ldots$ as shown below. This is an example of what is called the *breadth-first search* technique.



Question 2

Proof that the positive rational numbers is countable. The following TM *M* enumerates the positive rational numbers.

M = "On input $\langle \rangle$:

1. Print out the positive rationals in the following order $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \dots$ using a *breadth-first search* as shown below."



Therefore, the positive rational numbers is a countable set.

(Note, in our enumeration above, we print out the same element multiple times. For example, 1/1 and 2/2 are the same number. Similarly, both 1/2 and 2/4 will be printed out. This doesn't invalidate our enumeration. It does not matter if we print out the same element multiple times; it is only important that we don't leave any out.)

(Note also, in our breadth first search illustration above we have left out 0 from both the horizontal and vertical axis. If you had 0 in the vertical axis, it would not be a disaster, because you would just include 0/1, 0/2, etc. in your enumeration. It is important, however, not to have 0 in the horizontal axis because 1/0, 2/0, 3/0, etc., are not rationals.)