NUI MAYNOOTH

Ollscoil na hÉireann Má Nuad

OLLSCOIL NA hÉIREANN MÁ NUAD

NATIONAL UNIVERSITY OF IRELAND MAYNOOTH

SEMESTER 1
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COMPUTATION AND COMPLEXITY

PAPER CS370

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Time allowed: 2 hours

Answer ***three*** questions

All questions carry equal marks

1. (a) The following partial Turing machine M claims to recognise the language $L = \{w1w : w \in \{a,b\}^*\}$. However, four rows are missing that stop this Turing machine from working correctly. The start state is 00. To accept a word, M goes into state 99. The symbol '–' denotes a blank. [8 marks]

S_i	R	S_f	W	M
00	1	98	1	R
98	–	99	–	R
00	a	01	–	R
00	b	02	–	R
01	a	01	a	R
01	b	01	b	R
01	1	03	1	R
02	a	02	a	R
02	b	02	b	R
03	a	05	x	L
04	x	04	x	R
04	b	05	x	L
05	a	05	a	L
05	b	05	b	L
05	x	05	x	L
05	–	00	–	R

State the four missing rows.

(b) Specify new rows and new states (if necessary) that will convert this recogniser Turing machine M for L into a decider Turing machine M' for L. [4 marks]

(c) In the worst case, for an input word of length N, what is the maximum number of timesteps required for M' to decide the input word, and the maximum number of tape cells written to by M' to decide the word. Calculate these complexities exactly. [4 marks]

(d) Let the HITTINGSET problem be defined as follows. Given a finite set of subsets $A_1, A_2, \ldots, A_n$ of a finite set A, is there another subset of A containing at most k elements that has a nonempty intersection with each $A_1, A_2, \ldots, A_n$? Prove that HITTINGSET is in $\mathcal{NP}$. [5 marks]

(e) What are the steps required in proving a problem is $\mathcal{NP}$-complete. [4 marks]

2. (a) Prove that the set of words over any finite alphabet is countable. [5 marks]

(b) Prove that the set of languages over any finite alphabet is uncountable. [10 marks]

(c) How could you use a reduction to prove nonmembership of a class? [5 marks]

(d) What would be the implications if a Turing machine solving an $\mathcal{NP}$-complete problem was found to have (i) an exponential upper bound, or (ii) an exponential lower bound? [5 marks]

3. (a) What languages are in $\mathcal{NP}$? Give an example of four $\mathcal{NP}$ languages with different complexities. [8 marks]

(b) State three undecidable properties of Turing machines. [5 marks]

(c) Figure 1 on page 2 illustrates the space of languages 2^{Σ^*} for some finite Σ, where $|\Sigma| > 1$. Place each of the following languages, **and** its complement, in its appropriate place in this space. [12 marks]

i. $\text{MEM}_\text{J} = \{\langle J_1, J_2\rangle : J_1$ and J_2 are Java functions without arguments that require the same minimum amount of memory to run without crashing$\}$

ii. $\text{AT}_\text{TM} = \{\langle M, w\rangle : M$ is a Turing machine and w is a word and M accepts $w\}$

iii. $\text{NEM}_\text{TM} = \{\langle M\rangle : M$ is a Turing machine and $L(M) \neq \emptyset\}$

iv. $\text{NEQ}_\text{J} = \{\langle J_1, J_2\rangle : J_1$ and J_2 are Java functions that recognise different languages$\}$

v. $\text{SON}_\text{TM} = \{S : S$ is a binary string (strings are finite by default) and S contains at least three '1's$\}$

vi. $\text{SE}_\text{J} = \{\langle J_1, J_2\rangle : J_1$ and J_2 are Java functions that contain syntax errors$\}$

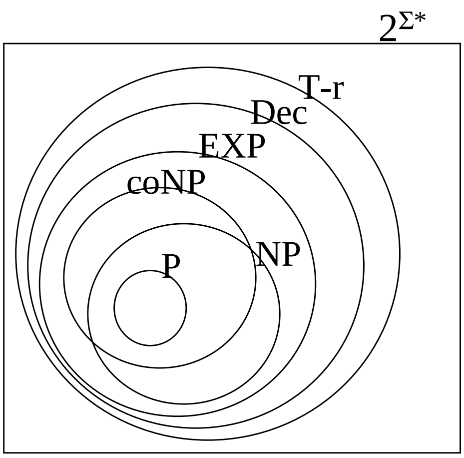

Figure 1: Illustration of the space of languages for question 3c: EXP denotes EXPTIME, Dec denotes the decidable languages (sometimes called the recursive languages), and T-r denotes the Turing-recognisable languages (sometimes called the recursively enumerable languages).

4. Let the language VAREQ_J be defined as $\text{VAREQ}_\text{J} = \{\langle J, a, b\rangle : J$ is a Java program, a and b are integer variables declared in J, and when J is run a and b have the same value at least once$\}$. You are given that HALT_J is undecidable. HALT_J is defined $\text{HALT}_\text{J} = \{\langle J, y\rangle : y \in \mathbb{Z}, J$ is a Java function that takes an integer argument and makes no function calls other than `System.out.Print()`, and J halts on input $y\}$.

   (a) Prove that VAREQ_J is undecidable. You may answer this question by assigning a name, mathematical construct, or piece of pseudocode to each of the numbered blanks in the proof template in Figure 2 on page 3. Where blanks have the same number, this denotes their contents will be the same. Alternatively, you can choose to ignore the template and construct your own proof from scratch. [15 marks]

   (b) Prove that VAREQ_J is Turing recognisable or prove that it is not Turing recognisable. [5 marks]

   (c) Give a definition of the language $\overline{\text{VAREQ}_\text{J}}$ (the complement of VAREQ_J). Prove that $\overline{\text{VAREQ}_\text{J}}$ is Turing recognisable or prove that it is not Turing recognisable. You may reuse your solutions to parts (a) and (b) in your proof. [5 marks]

Proof. We will use a mapping reduction to prove the reduction __1__. Assume that __2__ is decidable. The function f that maps instances of __3__ to instances of __4__ is performed by Turing machine F given by the following pseudocode.

$F =$ "On input $\langle$__5__$\rangle$:
  1. Construct the following M' given by the following pseudocode.
     $M' =$ "__6__"
  2. Output $\langle$__7__$\rangle$"

Now, $\langle$__7__$\rangle$ is an element of __8__ iff $\langle$__5__$\rangle$ is an element of __9__. So using f and the assumption that __2__ is decidable, we can decide __10__. A contradiction. Therefore, __2__ is undecidable. (This also means that the complement of __2__ is undecidable; the complement of any undecidable language is itself undecidable.)

Figure 2: Proof template for question 4a.