NUI MAYNOOTH

Ollscoil na hÉireann Má Nuad

OLLSCOIL NA hÉIREANN MÁ NUAD

THE NATIONAL UNIVERSITY OF IRELAND MAYNOOTH

SEMESTER 1
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DISCRETE STRUCTURES 1

PAPER CS151

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Time allowed: 2 hours

Answer ***three*** questions

All questions carry equal marks

1. (a) Given the following propositions P, Q, and R, [6 marks]

P : $\forall x, x$ is odd
Q : $\exists x, x + 1 = 1$
R : $\forall x$, if $2x$ is even then x is even

determine the truth values for each of the following compound propositions.

i. `NOT P AND (R OR NOT Q)`
ii. `(P OR Q) IF AND ONLY IF (R AND NOT P)`
iii. `NOT (IF P THEN (Q AND NOT R))`

(b) Using truth tables, prove the truth or falsity of each of the following equivalence statements, where $\neg A$ means "`not` A," and $\equiv$ means "is equivalent to." [4 marks]

i. $\neg(A \to B) \equiv \neg A \vee B$
ii. $A \to B \equiv (A \wedge \neg B) \to \texttt{False}$

(c) Prove each of the following propositions. Clearly state the proof strategy used in your solution. [15 marks]

i. If n^2 is odd, then n is odd
ii. If $c|a$ and $c|b$, then $c|(am + bn)$ for any integers m and n
iii. There exists a prime p such that $2^p - 1$ is not a prime

2. (a) In a survey of 120 adult shoppers at a supermarket, the following facts were recorded. [6 marks]

- 46 shoppers had driven there
- 60 of the shoppers were women
- 43 of the women had a loyalty card, as did 40 of the men
- 26 of the women had driven there
- 12 of the women drivers had a loyalty card
- 34 non-driving men had a loyalty card

Answer each of the following questions.

i. How many non-driving men were in the survey?
ii. How many men drivers had a loyalty card?
iii. How many driving shoppers did not have a loyalty card?

(b) For each of the following relations $R \subset A \times B$, add pairs to turn the relations into functions. (The default notion of a function is that it is a total function.) You just need to state the pairs added, not the whole function. Let $A = \{a, b, c, d, e\}$ and let $B = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. [2 marks]

i. $R = \{(c, 2), (e, 7), (a, 3)\}$
ii. $R = \{(a, 1), (c, 3), (e, 4)\}$

(c) For each of the following two bijections, (i) give a definition for the range of the bijection, and (ii) give a definition for the inverse of the bijection. [8 marks]

i. $f : \mathbb{N} \to A, f(x) = 2x + 1$
ii. $g : \mathbb{N} \to A, g(x) = \begin{cases} -\frac{x+1}{2}, & \text{if } x \text{ is odd} \\ x/2, & \text{if } x \text{ is even} \end{cases}$

(d) Give a definition for the range of the composition $g \circ f$, where g and f were defined in question 2c. [3 marks]

(e) Answer each of the following parts. [6 marks]

i. If x and y are odd integers, then xy is an even integer. Prove this false using a counter example.
ii. If x and y are odd integers, then xy is an odd integer. Prove this true using the direct approach.
iii. What is wrong with the following proof for part 2(e)ii: "We showed in part 2(e)i by a counterexample that it is not true to say that if x and y are odd integers, then xy is an even integer. Therefore this proves that if x and y are odd integers, then xy is an odd integer."

3. (a) Write down each step in the evaluation of $f(20)$ where f has the following recursive definition. [4 marks]
$f(1) = 0$
$f(n) = f(\lceil n/3 \rceil) + n$

(b) Write a recursive definition for each of the following functions. [6 marks]

i. $f : \mathbb{N} \to \mathbb{N}, f(n) = 1(2) + 2(3) + 3(4) + \ldots + n(n+1)$

ii. f : `String` $\to$ `String` that replaces each occurrence of the letter a by zz in a string over the alphabet $\{a, b, c\}$

(c) Construct an inductive proof to show that the following statement is true for all natural numbers greater than zero. [6 marks]
$1^2 + 2^2 + 3^2 + \ldots + n^2 = (n(n+1)(2n+1))/6$

(d) Write a recursive definition that prints the even elements of a list of integers. You may assume the existence of a `print` procedure that prints a single integer. Provide an inductive proof to show that the function definition is correct for all input lists. [9 marks]

4. (a) Write out the elements in the power set of $\{\emptyset, a, \{b\}, \{\emptyset, a\}\}$. [4 marks]

(b) Let $A = \{(a, b) : a, b \in S, a|b\}$, let $B = \{(a, b) : a, b \in S, a \leq b\}$, and let $S = \{1, 2, 3, 4\}$. [6 marks]

i. Prove that $B \not\subset A$.

ii. Prove that $A \subset B$.

(c) Let $P(x)$ denote the statement "x has a sweet tooth" and let $Q(x)$ denote the statement "x likes chocolate." The domain is the set of all people. Write each of the following propositions in English. [9 marks]

i. $\exists x(P(x) \land Q(x))$

ii. $\forall x(P(x) \lor Q(x))$

iii. $\forall x(P(x) \to Q(x))$

(d) Formalise each of the following English sentences where $P(x)$ denotes the statement "x is a swan" and $C(x, r)$ denotes the statement "x has colour r." [6 marks]

i. All swans are white

ii. Not all swans are white

iii. If some swan is blue then not all swans are white

Rules, Definitions, and Theorems that can be applied to any question

Not ($\overline{P}$, $\neg P$) truth table:

P	$\neg P$
T	F
F	T

And truth table:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Or truth table:

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Implication ($P \rightarrow Q$) truth table:

P	Q	if P then Q
T	T	T
T	F	F
F	T	T
F	F	T

If and only if (iff) truth table:

P	Q	P iff Q
T	T	T
T	F	F
F	T	F
F	F	T

Equivalence laws

not (A and B) $\equiv$ (not A) or (not B)
not (A or B) $\equiv$ (not A) and (not B)
A and (B or C) $\equiv$ (A and B) or (A and C)
A or (B and C) $\equiv$ (A or B) and (A or C)
if A then B $\equiv$ if not B then not A
if A then B $\equiv$ (not A) or B

Divisibility and prime numbers

$\mathbb{Z}$ is the set of whole numbers (includes negative numbers and 0) called integers.
$\mathbb{N}$ is the set of nonnegative whole numbers (includes 0) called naturals.
$\mathbb{R}$ is the set of real numbers.
The set of prime numbers is $\{x : x \in \mathbb{N}, x$ has exactly two factors: 1 and $x\}$. Therefore, 2 is the smallest prime number.
Integer d divides integer n (written $d|n$) if $d \neq 0$ and there is a $k \in \mathbb{Z}$ such that $n = dk$.

Proof techniques

- Proof by exhaustive techniques. To prove a statement is true by exhaustive checking one must prove that the statement is true for every possible value.
- Proof by counter example. To prove a statement false by a counter example one simply finds a single value for which the statement is false.
- Conditional proof (direct proof). One uses a conditional proof if one is asked to prove a statement of the form "if A then B." It requires one to first assume that A is true. Then one makes a statement consisting of A and any other known facts. If using the valid rules of logic one can derive B from this statement then this proves that the "if A then B" statement must be true.
- Conditional proof (proving the contrapositive). One might also prove a statement of the form "if A then B" by proving the contrapositive: proving that "if not B then not A." Here one would assume that B is false and then continue as in a direct proof to derive that A is false.
- Proof by contradiction. A proof by contradiction would involve one assuming the statement is false and then using this fact and any other known facts to derive a contradiction (i.e. such as deriving that an integer is both even and odd, or deriving that an element is both in and out of a particular set).
- If and only if proof. Proving a statement "A if and only if B," sometimes abbreviated to "A iff B," would require one to prove both "if A then B" and "if B then A."

Sets and tuples

A set A is a subset of B, written $A \subset B$, if for every $x \in A$ it is true that $x \in B$.
For every set A, both $\emptyset \subset A$ and $A \subset A$ are true.
A set A is equal to B, written $A = B$, if $A \subset B$ and $B \subset A$.
A set A is a proper subset of B if $A \subset B$ and $A \neq B$.
The union of two sets A and B is $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
The intersection of two sets A and B is $A \cap B = \{x : x \in A \text{ and } x \in B\}$.
The difference of two sets A and B is $A - B = \{x : x \in A \text{ and } x \notin B\}$.
The power set 2^A [sometimes written power(A)] of a set A is $\{x : x \subset A\}$.
A tuple is an ordered collection of objects that can contain duplicates.
Tuples are written using parentheses $(\ldots)$ rather than the braces $\{\ldots\}$ used for sets.
The cross product of two sets A and B is $A \times B = \{(a, b) : a \in A, b \in B\}$.

Functions

$f : A \to B$ is a function called f mapping elements from domain A to co-domain B.
The range of $f : A \to B$ is $\{f(a) : a \in A\}$.
$f : A \to B$ is equal to $g : A \to B$ if $f(a) = g(a)$ for all $a \in A$.
abs: $\mathbb{R} \to \mathbb{R}$ is defined abs$(x) =$ if $x \geq 0$ then x else $-x$.
floor: $\mathbb{R} \to \mathbb{Z}$, written $\lfloor x \rfloor$, is defined as the largest integer $\leq x$.
ceiling: $\mathbb{R} \to \mathbb{Z}$, written $\lceil x \rceil$, is defined as the smallest integer $\geq x$.
mod: $\mathbb{Z} \times (\mathbb{N} - \{0\}) \to \mathbb{N}$ is defined mod$(a, b) = a - b\lfloor a/b \rfloor$.