Ollscoil Mhá Nuad
Maynooth University

SEMESTER 2
2022-2023

CS605
The Mathematics and Theory of Computer Science

Prof. O. Conlan, Dr. J. Timoney, Prof. T. Naughton

Time allowed: 3 hours

Answer at least ***seven*** questions

Your mark will be based on your best ***seven*** answers

All questions carry equal marks

Instructions

	Yes	No	N/A
Formulae and Tables book allowed *(i.e. available on request)*		X	
Formulae and Tables book required *(i.e. distributed prior to exam commencing)*		X	
Statistics Tables and Formulae allowed *(i.e. available on request)*		X	
Statistics Tables and Formulae required *(i.e. distributed prior to exam commencing)*		X	
Dictionary allowed *(supplied by the student)*		X	
Non-programmable calculator allowed		X	
Students required to write in and return the exam question paper		X	
One physical (paper) copy of textbook Michael Sipser, Introduction to the Theory of Computation, without annotations or extra pages *(supplied by the student)*	X		
Students required to sign the declaration page at the end of this document and return it in their answer booklet	X		

Dear student, by starting the examination, you confirm that you have read the instructions in the box below.

Additional material allowed: For this exam you are permitted to bring one physical (paper) copy of the textbook Michael Sipser, *Introduction to the Theory of Computation* (any edition), without annotations or extra pages. Please sign the declaration page at the end of this document and return it in your answer booklet.

[10 marks]

1 For each of the following languages, prove that it is context-free or prove that it is not context-free, where superscript R has its usual meaning.

(a) $\{w_1\#w_2\#...\#w_n : w_i \in \{a, b\}^* \text{ and } w_i = w_j \text{ for some } j > i\}$

(b) $\{w_1\#w_2\#...\#w_n : w_i \in \{a, b\}^* \text{ and } w_i = w_j^R \text{ for some } j > i\}$

[10 marks]

2 Let the language $\text{MEAN}_{\text{Java}}$ be defined as $\text{MEAN}_{\text{Java}}$ = {<*J*, *b*, *c*> : *J* is a Java program, *b* is an integer variable declared in *J*, *c* is an integer array declared in *J*, and throughout the execution of *J*, *b* has a value equal to the mean (average) of the integers in array *c*}. (Note, this means that the mean of the integers in array *c* is an integer. Also, “throughout the execution of” means “at all times during the execution of”.)

Prove that $\text{MEAN}_{\text{Java}}$ is undecidable. You are given that HALTS is undecidable. HALTS is defined as HALTS = {<*M, w*> : *M* is a Turing machine that halts on input *w*}. You can use the template in Figure 1 if you wish.

[10 marks]

3 Consider the language $\text{MEAN}_{\text{Java}}$ from the previous question.

(a) Prove that $\text{MEAN}_{\text{Java}}$ is Turing-recognisable, or prove that it is not Turing-recognisable. You can refer to your answer from the previous question if you wish.

(b) Give a definition of the language $\text{NMEAN}_{\text{Java}}$ (the complement of $\text{MEAN}_{\text{Java}}$). Prove that $\text{NMEAN}_{\text{Java}}$ is Turing-recognisable, or prove that it is not Turing-recognisable.

[10 marks]

4 Let E = {<M> : M is a Turing machine with an input alphabet of {a, b} and L(M) = $\emptyset$, i.e. M accepts no words}.

Prove that E is not Turing-recognisable. You are given that HALTS is undecidable. HALTS is defined as HALTS = {<M, w> : M is a Turing machine that halts on input w}. You can use the template in Figure 1 if you wish.

[10 marks]

5 Prove the following theorem for languages A and B: if $A \leq_m B$ and A is not Turing-recognisable, then B is not Turing-recognisable.

[10 marks]

6 The SUBSET-SUM-U problem is defined as

SUBSET-SUM-U = {<S, t> : $S = \{x_1, ..., x_m\}$, where each $x_i \in \{0\}^*$, where t $\in \{0\}^*$, and for some subset $\{y_1, ..., y_n\} \subseteq \{x_1, ..., x_m\}$, it is the case that $\Sigma|y_i| = |t|$ (i.e. the sum of the lengths of strings y_i equals the length of t) }.

As examples,
<{0, 00, 000}, 0000> $\in$ SUBSET-SUM-U, and
<{00, 000, 00000}, 0000> $\notin$ SUBSET-SUM-U.

Prove that SUBSET-SUM-U is in $\mathcal{NP}$.

[10 marks]

7 Consider the language SUBSET-SUM-U from the previous question.

(a) Prove that SUBSET-SUM-U is regular or prove that it is not regular.

(b) Prove that SUBSET-SUM-U is context-free or prove that it is not context-free.

[10 marks]

8 The HITTING-SET-0 problem is defined as HITTING-SET-0 = {<T, k> : $T = \{S_1, ..., S_n\}$ is a system of sets and k is an integer, where each set $S_i \subseteq X$ is a subset of an underlying set X = $\{x_1, ..., x_m\}$, where each $x_i \in \{0\}^*$ is an integer represented in unary notation, and there exists another subset $H \subseteq X$ of size k that has a nonzero intersection with each $S_0, ..., S_n$}.

Prove that HITTING-SET-0 is $\mathcal{NP}$-complete. You are given that 3-SAT is $\mathcal{NP}$-complete.

Proof. We will use a mapping reduction to prove the reduction $\underline{\quad 1 \quad}$. Assume that $\underline{\quad 2 \quad}$ is decidable. The function f that maps instances of $\underline{\quad 3 \quad}$ to instances of $\underline{\quad 4 \quad}$ is performed by TM F given by the following pseudocode.

$F =$ "On input $\langle \underline{\quad 5 \quad} \rangle$:

1. Construct the following M' given by the following pseudocode.

    $M' =$ "$\underline{\quad 6 \quad}$"

2. Output $\langle \underline{\quad 7 \quad} \rangle$"

Now, $\langle \underline{\quad 7 \quad} \rangle$ is an element of $\underline{\quad 8 \quad}$ iff $\langle \underline{\quad 5 \quad} \rangle$ is an element of $\underline{\quad 9 \quad}$. So using f and the assumption that $\underline{\quad 2 \quad}$ is decidable, we can decide $\underline{\quad 10 \quad}$. A contradiction. Therefore, $\underline{\quad 2 \quad}$ is undecidable. (This also means that the complement of $\underline{\quad 2 \quad}$ is undecidable; the complement of any undecidable language is itself undecidable.)

Figure 1. Proof template that can be used if you wish. Where blanks have the same number, this denotes that their contents will be the same.

OLLSCOIL NA hÉIREANN MÁ NUAD

THE NATIONAL UNIVERSITY OF IRELAND MAYNOOTH

SUMMER 2023 EXAMINATION

CS605

The Mathematics and Theory of Computer Science

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Declaration

To be signed by each student and returned in their answer booklet at the end of the examination

i. I have searched through my copy of M. Sipser, *Introduction to the Theory of Computation*, any edition, (the Sipser book) and it does not contain any extra pages or annotations (except for annotations that correct minor typographical errors).

ii. I understand that by failing to notify an invigilator of any annotations or extra pages in my copies of the Sipser book, I will receive a mark of zero in this examination. This does not affect any further disciplinary actions that the University may wish to take.

iii. I understand also that directly copying large amounts of material from the Sipser books without substantially tailoring it to the question asked may result in a mark of zero.

Print name ____________________ Student number __________________

Signed ___________________________ Date ______________________