Ollscoil Mhá Nuad
Maynooth University

SEMESTER 2
2021-2022

CS605
The Mathematics and Theory of Computer Science

Prof. O. Conlan, Dr. J. Timoney, Prof. T. Naughton

Time allowed: 3 hours

Answer at least ***seven*** questions

Your mark will be based on your best ***seven*** answers

All questions carry equal marks

Instructions

	Yes	No	N/A
Formulae and Tables book allowed *(i.e. available on request)*		X	
Formulae and Tables book required *(i.e. distributed prior to exam commencing)*		X	
Statistics Tables and Formulae allowed *(i.e. available on request)*		X	
Statistics Tables and Formulae required *(i.e. distributed prior to exam commencing)*		X	
Dictionary allowed *(supplied by the student)*		X	
Non-programmable calculator allowed		X	
Students required to write in and return the exam question paper	X		
One physical (paper) copy of textbook Michael Sipser, Introduction to the Theory of Computation, without annotations or extra pages *(supplied by the student)*	X		
Students required to sign the declaration page at the end of this document and return it in their answer booklet	X		

Dear student, by starting the examination, you confirm that you have read the instructions in the box below.

Additional material allowed: For this exam you are permitted to bring one physical (paper) copy of the textbook Michael Sipser, *Introduction to the Theory of Computation* (any edition), without annotations or extra pages. Please sign the declaration page at the end of this document and return it in your answer booklet.

[10 marks]

1 For each of the following languages, prove that it is regular or prove that it is not regular

(a) $\{w_1\#w_2\#...\#w_{3n} : w_i \in \{\text{a}\}^*,\ n > 0, \text{ and } w_i = \text{a for each } i \bmod 3 = 0\}$

(b) $\{w_1\#w_2\#...\#w_{2n} : w_i \in \{\text{a, b}\}^*,\ n > 0, \text{ and } w_1 = w_2\}$

(c) $\{w_1w_2...w_{2n} : w_i \in \{\text{a, b}\}^*,\ n > 0\}$

[10 marks]

2 For each of the following languages, prove that it is context-free or prove that it is not context-free, where superscript R has its usual meaning.

(a) $\{w_1\#w_2\#...\#w_{2n} : w_i \in \{\text{a}\}^*,\ n > 0, \text{ and } w_i = w_{i+1} \text{ for each odd integer } i\}$

(b) $\{w_1w_2...w_{2n} : w_i \in \{\text{a, b}\}^*,\ n > 0, \text{ and } w_i = w_{i+1}{}^{\text{R}} \text{ for each odd integer } i\}$

(c) $\{w_1\#w_2\#...\#w_n : w_i \in \{\text{a, b}\}^*,\ n > 0, \text{ and } w_1 = w_j^{\text{R}} \text{ for each } j > 1\}$

[10 marks]

3 Consider the following sextuple of language classes: (regular, context-free, Turing-recognisable, decidable, $\mathcal{P}$, $\mathcal{NP}$).

Each language can be associated with a binary sextuple where symbol 1 denotes membership and 0 denotes nonmembership of the class in question. For example, if a language was in the first two classes and not in any of the others, it would be associated with the binary sextuple (1, 1, 0, 0, 0, 0). State the binary sextuple associated with each of the following languages.

(a) The language L = {aabbaa, bbbbbb, ababbaba}.

(b) The language L = {w : $w \in$ {a, b}*, |w| is odd, w has exactly one b positioned exactly in the centre of the word}.

(c) The language L = {w : $w \in$ {a, b}*, |w| is odd, w has exactly one b}.

(d) The language of chess board configurations for which white can win.

[10 marks]

4 A software company is worried about their software product, written in Java, running out of memory because some Java Virtual Machine implementations require the maximum memory allowed to be stated in advance. Given their Java program and a memory limit in gigabytes, they want proof that the program will stay within that memory limit while the program is running.

The language associated with the problem is
MEMLIMIT = {<*J*, *m*> : *J* is a Java program and m is a non-negative integer, and when *J* is run it uses at most *m* gigabytes of memory}.

Prove that MEMLIMIT is undecidable. You are given that HALT = {<*M*> : *M* is a Turing machine that halts when it is run} is undecidable. You can use the template in Figure 1 if you wish.

[10 marks]

5 Consider the language from the previous question
MEMLIMIT = {<*J*, *m*> : *J* is a Java program and *m* is a non-negative integer, and when *J* is run it uses at most *m* gigabytes of memory}.

Answer both parts (a) and (b).

(a) Prove that MEMLIMIT is either ***not*** Turing recognisable, or ***not*** co-Turing recognisable, but cannot be both. You can refer to your answer from the previous question if you wish. [7 marks]

(b) In general, is it possible for a language be both ***not*** Turing recognisable and ***not*** co-Turing recognisable? Outline a proof of your answer. [3 marks]

[10 marks]

6 FINITE = {<*M*, *n*, *W*> : *M* is a set of finite automata, integer $n > 0$, *W* is a set of *n* words over {a, b}*, each $w \in W$ has length exactly *n*, and each $w \in W$ is accepted by each machine in *M*}. Prove that this language is in $\mathcal{P}$.

[10 marks]

7 QUARTER = {<*A*> : $A = \{x_1, ..., x_m\}$ is a set of natural numbers and there exists a subset $A' \subseteq A$ where the sum of the elements in *A*' is exactly a quarter of the sum of the elements in *A*}. Prove that this language is in $\mathcal{NP}$.

[10 marks]

8 A board game manufacturer in the future wishes to translate its instructions into as few languages as possible. Because of the high quality of automated natural language translation apps, some countries allow games to be sold with instructions in any of the set of *m* modern languages. However, some countries force each game manufacturer to have instructions in at least one of their official languages (each a subset of the *m* languages). The board game manufacturer wishes to translate its instructions into no more than *k* languages such that each of its *n* customer countries is satisfied with at least one of those languages. Prove that this problem is $\mathcal{NP}$-complete.

You are given only that the problem SATISFIABILITY (also known as SAT) is $\mathcal{NP}$-complete.

Proof. We will use a mapping reduction to prove the reduction $\underline{\quad 1 \quad}$. Assume that $\underline{\quad 2 \quad}$ is decidable. The function f that maps instances of $\underline{\quad 3 \quad}$ to instances of $\underline{\quad 4 \quad}$ is performed by TM F given by the following pseudocode.

$F =$ "On input $\langle \underline{\quad 5 \quad} \rangle$:

1. Construct the following M' given by the following pseudocode.

   $M' =$ "$\underline{\quad 6 \quad}$"

2. Output $\langle \underline{\quad 7 \quad} \rangle$"

Now, $\langle \underline{\quad 7 \quad} \rangle$ is an element of $\underline{\quad 8 \quad}$ iff $\langle \underline{\quad 5 \quad} \rangle$ is an element of $\underline{\quad 9 \quad}$. So using f and the assumption that $\underline{\quad 2 \quad}$ is decidable, we can decide $\underline{\quad 10 \quad}$. A contradiction. Therefore, $\underline{\quad 2 \quad}$ is undecidable. (This also means that the complement of $\underline{\quad 2 \quad}$ is undecidable; the complement of any undecidable language is itself undecidable.)

Figure 1. Proof template that can be used if you wish. Where blanks have the same number, this denotes that their contents will be the same.

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THE NATIONAL UNIVERSITY OF IRELAND MAYNOOTH

SUMMER 2022 EXAMINATION

CS605

The Mathematics and Theory of Computer Science

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Declaration

To be signed by each student and returned in their answer booklet at the end of the examination

i. I have searched through my copy of M. Sipser, *Introduction to the Theory of Computation*, any edition, (the Sipser book) and it does not contain any extra pages or annotations (except for annotations that correct minor typographical errors).

ii. I understand that by failing to notify an invigilator of any annotations or extra pages in my copies of the Sipser book, I will receive a mark of zero in this examination. This does not affect any further disciplinary actions that the University may wish to take.

iii. I understand also that directly copying large amounts of material from the Sipser books without substantially tailoring it to the question asked may result in a mark of zero.

Print name ____________________ Student number ________________

Signed ___________________________ Date ______________________