Maynooth University
National University of Ireland Maynooth

OLLSCOIL NA hÉIREANN MÁ NUAD

THE NATIONAL UNIVERSITY OF IRELAND MAYNOOTH

SUMMER 2019 EXAMINATION

CS605

The Mathematics and Theory of Computer Science

Dr. R. Bond, Dr. J. Timoney, Prof. T. Naughton

Time allowed: 3 hours

Answer at least three questions

Your mark will be based on your best ***three*** answers

All questions carry equal marks

Additional material allowed:
Copy of Michael Sipser, *Introduction to the Theory of Computation* (any edition), without annotations or extra pages. Please sign the declaration page at the end of this document and return it in your answer booklet.

[25 marks]

1 (a) State whether each of the following is true or false. [6 marks]

i. $\emptyset \in \emptyset$

ii. $\emptyset = 2^{\emptyset}$

iii. $\{a, b\} \subseteq 2^{\{a, b, \{a, b\}\}}$

(b) Let Σ = {*a*, *b*, *c*} and let *L* = {*w* : *w* ∈ Σ*, |*w*| > 2 or *w* contains the same number of *a*s and *b*s}. [5 marks]
Write down the first five elements in the lexicographical ordering of *L*, where Σ has the usual alphabetical ordering (*a*, *b*, *c*).

(c) Let $\mathbb{R}$ be the set of real numbers. Let $\mathbb{Z}$ be the set of integers. Let $\mathbb{N}$ be the set of natural numbers. State whether each of the following sets is countable or not. [8 marks]

i. $2^{\mathbb{R}}$

ii. $2^{\mathbb{Z}}$

iii. $2^{\mathbb{N}}$

iv. $2^{\mathbb{R} \cap \mathbb{Z}}$

(d) Can you enumerate the set of all words over a finite alphabet? Prove your answer. [6 marks]

[25 marks]

2 (a) What is a polynomially-bounded (or polynomial-time) reduction? Illustrate with an example why these are of interest in computational complexity theory. [5 marks]

(b) You want to find an algorithmic solution for problem *A*. You know that *A* is in NP. [4 marks]
Should you look for an efficient algorithm for *A*? Explain.

(c) Is the problem of writing out the factorial of a number in unary NP-complete or NP-hard (e.g. *n*! = 111111 for *n* = 3)? Explain. [4 marks]

(d) A board game manufacturer in the future wishes to translate its instructions into as few languages as possible. Because of the high quality of automated natural language translation apps, some countries allow games to be sold with instructions in any of the set of *m* modern languages. However, some countries force each game manufacturer to have instructions in at least one of their official languages (each a subset of the *m* languages). The board game manufacturer wishes to translate its instructions into no more than *k* languages such that each of its *n* customer countries is satisfied with at least one of those languages. Prove that this problem is NP-complete. [12 marks]
You are given only that the problem SATISFIABILITY (also known as SAT) is NP-complete.

[25 marks]

3 A software company wants to improve the efficiency of function call parameter passing in its software products. In the existing code, there were many instances of objects being passed by value to functions that did not change these objects under any circumstances. In order to avoid the overhead of implicitly copying the objects when such functions are called it would be preferable to pass the objects by reference. The company is writing a software routine that takes the code of a function *F* as input and determines whether its arguments (currently passed by value) would be updated during the execution of *F* or not. If an argument would not be updated then it could be safely passed by reference without changing the semantics of the program. (Assume that functions only refer to variables passed as arguments or declared within the function body – i.e. there are no global variables to contend with).

The language associated with this problem is $\text{PASSBYREF}_{\text{MULTIARG}}$ = {<*F*, *A*, *B*, ...> : *F* is a function, *A*, *B*, ..., are arguments to *F*, and none of the arguments is modified when *F* is run}.

Consider also a restricted version of the problem where *F* contains only one argument *A* and there are no other function calls within the body of *F*. This language associated with this problem is $\text{PASSBYREF}_{\text{ONEARG}}$ = {<*F*, *A*> : *F* is a function that does not call any other functions, *A* is the only argument of *F*, and *A* is not modified when *F* is run}.

(a) Prove that $\text{PASSBYREF}_{\text{ONEARG}}$ is Turing-recognisable or prove that it is not Turing-recognisable. [6 marks]

(b) Prove that $\text{PASSBYREF}_{\text{ONEARG}}$ is decidable or prove that it is not decidable. You are given that HALTS = {<*M*> : *M* is a Turing machine that halts on all inputs} is undecidable. Use the template in Figure 1 if you wish. [13 marks]

(c) Considering the more general language $\text{PASSBYREF}_{\text{MULTIARG}}$, prove the decidability or undecidability of this language. You can refer to your solution to parts (a) and (b) in your proof. [6 marks]

[25 marks]

4 (a) Consider the following sextuple of language classes: (regular, context-free, Turing-recognisable, decidable, P, NP).
Each language can be associated with a binary sextuple where symbol 1 denotes membership and 0 denotes nonmembership of the class in question. For example, if a language was in the first two classes and not in any of the others, it would be associated with the binary sextuple (1, 1, 0, 0, 0, 0). State the binary sextuple associated with each of the following languages.

i. The language $L = \{a, aa, ab\}$. [1 mark]

ii. The language $L = \{w : w \in \{a, b\}^*, |w|$ is odd, w has exactly one b positioned exactly in the centre of the word$\}$. [3 marks]

iii. The language $L = \{w : w \in \{a, b\}^*, |w|$ is odd, w has exactly one $b\}$. [3 marks]

iv. The language of chess board configurations for which white can win. [3 marks]

v. The language of Turing machines that contain a state 01. [3 marks]

vi. The language of Turing machines that go into a state 01. [3 marks]

vii. The language of set systems that can be hit by a set of cardinality 12 (where 'hit' is a technical term defined in the well-known Hitting Set problem). [3 marks]

viii. The language of graphs that have a tour that visits each vertex at least once. [3 marks]

(b) State three binary sextuples that cannot exist, and indicate where the contradiction occurs in each case. [3 marks]

Proof. We will use a mapping reduction to prove the reduction $\underline{\quad 1 \quad}$. Assume that $\underline{\quad 2 \quad}$ is decidable. The function f that maps instances of $\underline{\quad 3 \quad}$ to instances of $\underline{\quad 4 \quad}$ is performed by TM F given by the following pseudocode.

$F =$ "On input $\langle \underline{\quad 5 \quad} \rangle$:

1. Construct the following M' given by the following pseudocode.
   $M' =$ "$\underline{\quad 6 \quad}$"
2. Output $\langle \underline{\quad 7 \quad} \rangle$"

Now, $\langle \underline{\quad 7 \quad} \rangle$ is an element of $\underline{\quad 8 \quad}$ iff $\langle \underline{\quad 5 \quad} \rangle$ is an element of $\underline{\quad 9 \quad}$. So using f and the assumption that $\underline{\quad 2 \quad}$ is decidable, we can decide $\underline{\quad 10 \quad}$. A contradiction. Therefore, $\underline{\quad 2 \quad}$ is undecidable. (This also means that the complement of $\underline{\quad 2 \quad}$ is undecidable; the complement of any undecidable language is itself undecidable.)

Figure 1. Proof template that can be used if you wish. Where blanks have the same number, this denotes that their contents will be the same.

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Declaration

To be signed by each student and returned in their answer booklet
at the end of the examination

i. I have searched through my copy of M. Sipser, *Introduction to the Theory of Computation*, any edition, (the Sipser book) and it does not contain any extra pages or annotations (except for annotations that correct minor typographical errors).

ii. I understand that by failing to notify an invigilator of any annotations or extra pages in my copies of the Sipser book, I will receive a mark of zero in this examination. This does not affect any further disciplinary actions that the University may wish to take.

iii. I understand also that directly copying large amounts of material from the Sipser books without substantially tailoring it to the question asked may result in a mark of zero.

Print name ______________________ Student number ___________________

Signed ______________________________ Date _________________________