OLLSCOIL NA hÉIREANN MÁ NUAD

THE NATIONAL UNIVERSITY OF IRELAND MAYNOOTH

SUMMER 2018 EXAMINATION

CS605

The Mathematics and Theory of Computer Science

Dr. L. Beus-Dukic, Prof. A. Winstanley, Prof. T. Naughton

Time allowed: 3 hours

Answer at least three questions
Your mark will be based on your best ***three*** answers

All questions carry equal marks

Additional material allowed:
Copy of Michael Sipser, *Introduction to the Theory of Computation* (any edition), without annotations or extra pages. Please sign and return the declaration page at the end of this document.

[25 marks]

1 (a) Prove that the set of regular languages is closed under intersection. You can assume that it is known that the regular languages are closed under union and complement. [4 marks]

(b) Let the language $\text{PRINTDIF}_{\text{Java}}$ be defined as $\text{PRINTDIF}_{\text{Java}}$ = {<*J*> : *J* is a Java program and throughout the execution of *J*, each executed call to `print()`(after the first) prints a different string from the previous call to `print()`}. (For example, [13 marks]

```
void main(void){
  print("a"); print("b"); print("a");
}
```

is in the language because the previous call to `print` is different each time.) You are given that HALT is undecidable. HALT is defined as HALT = {<*M*> : *M* is a Turing machine that halts on all inputs}.

Prove that $\text{PRINTDIF}_{\text{Java}}$ is undecidable. You may answer this question by assigning a name, mathematical construct, or piece of pseudocode to each of the numbered blanks in the proof template in Figure 1. Where blanks have the same number, this denotes that their contents will be the same. Alternatively, you can choose to ignore the template and construct your own proof from scratch.

(c) Prove that $\text{PRINTDIF}_{\text{Java}}$ is Turing-recognisable or prove that it is not Turing-recognisable. [4 marks]

(d) Give a definition of the language $\text{NPRINTDIF}_{\text{Java}}$ (the complement of $\text{PRINTDIF}_{\text{Java}}$). Prove that $\text{NPRINTDIF}_{\text{Java}}$ is Turing-recognisable or prove that it is not Turing-recognisable. [4 marks]

Proof. We will use a mapping reduction to prove the reduction __1__. Assume that __2__ is decidable. The function f that maps instances of __3__ to instances of __4__ is performed by TM F given by the following pseudocode.

$F =$ "On input $\langle$ __5__ $\rangle$:

1. Construct the following M' given by the following pseudocode.

   $M' =$ " __6__ "

2. Output $\langle$ __7__ $\rangle$"

Now, $\langle$ __7__ $\rangle$ is an element of __8__ iff $\langle$ __5__ $\rangle$ is an element of __9__. So using f and the assumption that __2__ is decidable, we can decide __10__. A contradiction. Therefore, __2__ is undecidable. (This also means that the complement of __2__ is undecidable; the complement of any undecidable language is itself undecidable.)

Figure 1. Proof template that can be used if you wish. Where blanks have the same number, this denotes that their contents will be the same.

[25 marks]

2 (a) For each of the following languages, prove that it is context-free or prove that it is not context-free, where superscript R has its usual meaning. [21 marks]

i. $\{w_1\#w_2\#...\#w_n : w_i \in \{a\}^* \text{ and } w_i = w_j \text{ for some } i \text{ and } j\}$

ii. $\{w_1\#w_2\#...\#w_{2n} : w_i \in \{a\}^*, n > 0$, and
$w_i = w_{i+1}$ for each $i \in \{o : o > 0, o \text{ is odd}\}\}$

iii. $\{w_1w_2...w_{2n} : w_i \in \{a, b\}^*, n > 0$, and
$w_i = w_{i+1}^R$ for each $i \in \{o : o > 0, o \text{ is odd}\}\}$

iv. $\{w_1\#w_2\#...\#w_n : w_i \in \{a, b\}^*, n > 0, \text{ and } w_1 = w_j^R \text{ for each } j > 1\}$

v. $\{w\#a_1v_1\#a_2v_2...\#a_nv_n : w \in \{a, b\}^*, v_i \in \{a\}^*, a_i \in \{a, b\}$, and
$w = a_1a_2...a_n\}$

(b) Consider the following two languages, where L has its usual meaning. Comment on the whether each one is Turing-recognisable, decidable, both, or neither. [4 marks]

i. $\{<M_1, M_2> : M_1, M_2 \text{ are finite automata and } L(M_1) = L(M_2)\}$

ii. $\{<M_1, M_2> : M_1, M_2 \text{ are Turing machines and } L(M_1) = L(M_2)\}$

[25 marks]

3 (a) Let $\text{SUBSET}_{FA} = \{<A, B> : A \text{ and } B \text{ are finite automata and } L(A) \subseteq L(B)\}$, where L has its usual meaning. Prove that SUBSET_{FA} is decidable. You can assume in your proof the existence of a Turing machine T that decides the language $\text{EMPTY}_{FA} = \{<A> : A \text{ is a finite automaton and } L(A) = \emptyset\}$. [10 marks]

(b) For each exam x being held in the University, you are given a list of students E_x attending that exam. Students may attend multiple exams on multiple days. The Students' Union wishes to find out if the noise level was too high during any of the exams, and has asked you to select k students from the set of all students $S = E_0 \cup E_1 \cup ...$ such that each exam was attended by at least one of the k students. Prove that the problem you are asked to solve is $\mathcal{NP}$-complete. [15 marks]

[25 marks]

4 For each of the following languages, prove that it is in $\mathcal{NP}$ or prove that it is not in $\mathcal{NP}$. [25 marks]

(a) PARTITION = $\{<A> : A = \{x_1, ..., x_m\}$ is a set of natural numbers and there exists a subset $A' \subseteq A$ where $A' = \{y_1, ..., y_m\}$ such that $\Sigma_i y_i = ½\Sigma_i x_i \}$.

(b) 3D-MATCHING = $\{<A, k> : A$ is a set of triples $A \subseteq (X \times Y \times Z)$ where X, Y, Z are disjoint (they share no elements) finite sets each containing k elements, and there exists a subset $A' \subseteq A$ also of size k with the property that no element appears in multiple triples in $A' \}$.

(c) TRAVELLING-SALESPLAN = $\{<V, s, t> : V$ defines the vertices of a fully-connected undirected graph, $s \in V$ is called the starting vertex, and t is the total number of ways that one can tour (visit each of the vertices exactly once) the graph starting from $s \}$.

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Declaration

To be signed by the student and collected by an invigilator at the beginning of the examination

i. I have searched through my copy of M. Sipser, *Introduction to the Theory of Computation*, any edition, (the Sipser book) and it does not contain any extra pages or annotations (except for annotations that correct minor typographical errors).

ii. I understand that by failing to notify an invigilator of any annotations or extra pages in my copies of the Sipser book, I will receive a mark of zero in this examination. This does not affect any further disciplinary actions that the University may wish to take.

iii. I understand also that directly copying large amounts of material from the Sipser books without substantially tailoring it to the question asked may result in a mark of zero.

Print name ______________________ Student number ___________________

Signed ______________________________ Date _________________________