Maynooth University
National University of Ireland Maynooth

OLLSCOIL NA hÉIREANN MÁ NUAD

THE NATIONAL UNIVERSITY OF IRELAND MAYNOOTH

SUMMER 2017 EXAMINATION

CS605

The Mathematics and Theory of Computer Science

Dr. L. Beus-Dukic, Prof. A. Winstanley, Prof. T. Naughton

Time allowed: 3 hours

Answer at least three questions
Your mark will be based on your best ***three*** answers

All questions carry equal marks

Additional material allowed:
Copy of Michael Sipser, *Introduction to the Theory of Computation* (any edition), without annotations or extra pages. Please sign and return the declaration page at the end of this document.

[25 marks]

1 Question 1 consists of thirteen multiple choice questions (MCQs) labelled (1.I) to (1.XIII). For each MCQ you will get 2 marks for choosing the correct answer, -0.5 (minus 0.5) for choosing an incorrect answer, and 0 for not choosing an answer. The maximum marks possible for Question 1 is 25 marks.

1.I You are given Turing machine $T_1 = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject} \rangle$ = $\langle$ {00, 01, 02, 99}, {0}, {0, 1, –}, δ, 00, 99, 50 $\rangle$ that operates on unary numbers, where δ is

q	*s*	*q'*	*s'*	*m*
00	0	01	1	R
00	–	99	–	R
01	0	01	0	R
01	–	02	–	L
02	0	02	0	L
02	1	00	1	R

where – is the special blank symbol, and where 'R' denotes 'move right', 'L' denotes 'move left', and 'S' denotes 'stay in this tape square'.

What is the running time of T_1 for an input length of N?

(a) $2N^2 + 2$
(b) $2N + 3$
(c) $N^2 + 2N + 2$
(d) $2N^2 + 2N + 2$
(e) $2N + 6$

1.II You are given Turing machine T_1 from the prevoius question. What would be the halting state of the machine for an invalid input "<u>0</u>010" (without quotes) where each option below is given in the form STATE : TAPE SYMBOLS, and an underscore "_" denotes the position of the tape head.

(a) 01: 11<u>1</u>0
(b) 01: 10<u>1</u>0
(c) 00: <u>0</u>010
(d) 01: 1111_
(e) 00: 10<u>1</u>0

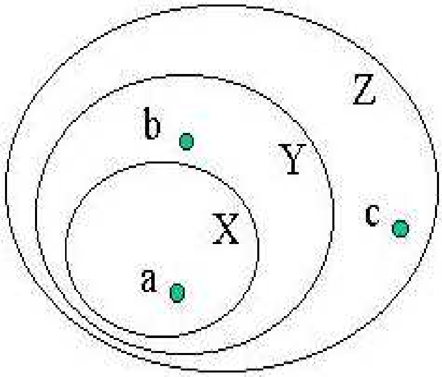
1.III "All machines have equal power (in terms of computability)." This statement

(a) is a direct result of the Church-Turing thesis
(b) is generally accepted, but cannot be proved true (it is a thesis rather than a theorem)
(c) concerns only those machines that recognise languages
(d) all of the above
(e) none of the above

1.IV If the set of all words over alphabet Σ is countable then

(a) each language over Σ must be countable
(b) at least one language over Σ is uncountable
(c) each language over Σ is finite
(d) no language over Σ is finite
(e) none of the above

1.V Given the diagram below, depicting sets $X \subseteq Y \subseteq Z$ and elements $a \in X$, $b \in Y$, $c \in Z$, which of the following statements is true?



(a) a is Y-hard
(b) a is Z-complete
(c) b is Z-complete
(d) b is X-complete
(e) none of the above

1.VI If it is proved that a solution to a decidable problem cannot be verified in polynomial time then

(a) the solution cannot be verified in exponential time
(b) the problem is not Turing-recognisable
(c) the verification process is not decidable
(d) the problem is intractable
(e) none of the above

1.VII The problem of deciding whether two arbitrary finite automata are equivalent

(a) is decidable
(b) is Turing-recognisable but not decidable
(c) is undecidable
(d) is co-Turing-recognisable
(e) none of the above

1.VIII The set of *k*-tape Turing machines is not equivalent (in terms of computational power) to which of the following sets?

(a) the set of 1-tape Turing machines
(b) the set of 2-tape Turing machines
(c) the set of decidable languages
(d) the set of Turing-recognisable languages
(e) the set of Java programs

For the next three MCQs, consider the following sextuple of language classes: (regular, context-free, Turing-recognisable, decidable, P, NP).
Each language can be associated with a binary sextuple where symbol 1 denotes membership and 0 denotes nonmembership of the class in question. For example, if a language was in the first two classes and not in any of the others, it would be associated with the binary sextuple (1,1,0,0,0,0).

1.IX State the binary sextuple associated with the language $A \circ B$, where A and B are regular languages, and where the symbol $\circ$ means concatenation.

(a) (0, 0, 0, 0, 0, 0)
(b) (0, 0, 1, 1, 0, 1)
(c) (0, 0, 1, 1, 0, 0)
(d) (0, 1, 1, 1, 1, 1)
(e) none of the above

1.X State the binary sextuple associated with the language {<M> : M is a FA that accepts all words}.

(a) (1, 1, 1, 1, 1, 1)
(b) (1, 0, 0, 0, 1, 1)
(c) (0, 0, 1, 1, 0, 1)
(d) (0, 0, 1, 1, 1, 1)
(e) none of the above

1.XI State the binary sextuple associated with the language {<M> : M is a TM that accepts all words}.

(a) (0, 0, 0, 0, 0, 0)
(b) (0, 0, 1, 1, 0, 0)
(c) (0, 0, 1, 0, 0, 1)
(d) (0, 0, 1, 1, 0, 1)
(e) none of the above

1.XII Which of the following languages is not Turing-recognisable?

(a) the set of nondeterministic finite automata
(b) the set of deterministic pushdown automata
(c) the halting Turing machines
(d) the Turing machines
(e) none of the above

1.XIII Which of the following languages is not decidable?

(a) the set of pushdown automata that halt on input *e*
(b) the set of Turing machines with *N* or more states
(c) the set of Turing machines
(d) the set of halting Java programs
(e) the set of non-halting finite automata

[25 marks]

2 (a) Construct a Turing machine with exactly $aN^2 + b$ time complexity, where N is the length of the input and a and b are arbitrary constants that can have any integer value. You can give pseudocode (in the style of Sipser) or an explicit table of behaviour. [4 marks]

(b) Let the language $\text{MEAN}_{\text{Java}}$ be defined as $\text{MEAN}_{\text{Java}}$ = {<*J*, *b*, *c*> : *J* is a Java program, *b* is an integer variable declared in *J*, *c* is an integer array declared in *J*, and throughout the execution of *J*, *b* has a value equal to the mean (average) of the integers in array *c*}. (Note, this means that the mean of the integers in array *c* is an integer. Also, “throughout the execution of” means “at all times during the execution of”.) You are given that HALT is undecidable. HALT is defined as HALT = {<*M*> : *M* is a Turing machine that halts on all inputs}. [13 marks]
Prove that $\text{MEAN}_{\text{Java}}$ is undecidable. You may answer this question by assigning a name, mathematical construct, or piece of pseudocode to each of the numbered blanks in the proof template in Figure 1. Where blanks have the same number, this denotes that their contents will be the same. Alternatively, you can choose to ignore the template and construct your own proof from scratch.

(c) Prove that $\text{MEAN}_{\text{Java}}$ is Turing-recognisable or prove that it is not Turing-recognisable. [4 marks]

(d) Give a definition of the language $\text{NMEAN}_{\text{Java}}$ (the complement of $\text{MEAN}_{\text{Java}}$). Prove that $\text{NMEAN}_{\text{Java}}$ is Turing-recognisable or prove that it is not Turing-recognisable. [4 marks]

Proof. We will use a mapping reduction to prove the reduction __1__. Assume that __2__ is decidable. The function f that maps instances of __3__ to instances of __4__ is performed by TM F given by the following pseudocode.

F = “On input $\langle$ __5__ $\rangle$:
1. Construct the following M' given by the following pseudocode.
   M' = “ __6__ ”
2. Output $\langle$ __7__ $\rangle$”

Now, $\langle$ __7__ $\rangle$ is an element of __8__ iff $\langle$ __5__ $\rangle$ is an element of __9__. So using f and the assumption that __2__ is decidable, we can decide __10__. A contradiction. Therefore, __2__ is undecidable. (This also means that the complement of __2__ is undecidable; the complement of any undecidable language is itself undecidable.)

Figure 1. Proof template that can be used if you wish. Where blanks have the same number, this denotes that their contents will be the same.

[25 marks]

3 (a) For each of the following languages, prove that it is context-free or prove that it is not context-free, where superscript R has its usual meaning. [20 marks]

i. $\{w_1\#w_2\#...\#w_n : w_i \in \{a, b\}^*$ and $w_i = w_j$ for some $j > i\}$

ii. $\{w_1\#w_2\#...\#w_n : w_i \in \{a, b\}^*$ and $w_i = w_j^R$ for some $j > i\}$

iii. $\{w_1\#w_2\#...\#w_n : w_i \in \{a, b\}^*$ and $w_i = w_j$ for all i and $j\}$

(b) Let L = {<M> : M is a Turing machine with an input alphabet of {a, b} and L(M) = ∅, i.e. M accepts no words}. Prove that the complement of L is Turing-recognisable. [5 marks]

[25 marks]

4 (a) The SUBSET-SUM-0 problem is defined as [7 marks]

SUBSET-SUM-0 = {<S, t> : $S = \{x_1, ..., x_m\}$ is a set of numbers where each $x_i \in \{0\}^*$ is an integer represented in unary notation, and for some subset $\{y_1, ..., y_n\} \subseteq \{x_1, ..., x_m\}$, it is the case that $\Sigma y_i = t$ }.

Prove that SUBSET-SUM-0 is in $\mathcal{NP}$.

(b) The HITTING-SET-0 problem is defined as [7 marks]

HITTING-SET-0 = {<T, k> : $T = \{S_1, ..., S_n\}$ is a system of sets and k is an integer, where each set $S_i \subseteq X$ is a subset of an underlying set $X = \{x_1, ..., x_m\}$, where each $x_i \in \{0\}^*$ is an integer represented in unary notation, and there exists another subset $H \subseteq X$ of size k that has a nonzero intersection with each $S_0, ..., S_n$}.

Prove that HITTING-SET-0 is in $\mathcal{NP}$.

(c) You are given that 3-SAT is $\mathcal{NP}$-complete. Prove that HITTING-SET-0 is $\mathcal{NP}$-complete. You can use your answer to part (b) in your proof. [11 marks]

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Declaration
To be signed by the student and collected by an invigilator at the beginning of the examination

i. I have searched through my copy of M. Sipser, *Introduction to the Theory of Computation*, any edition, (the Sipser book) and it does not contain any extra pages or annotations (except for annotations that correct minor typographical errors).

ii. I understand that by failing to notify an invigilator of any annotations or extra pages in my copies of the Sipser book, I will receive a mark of zero in this examination. This does not affect any further disciplinary actions that the University may wish to take.

iii. I understand also that directly copying large amounts of material from the Sipser books without substantially tailoring it to the question asked may result in a mark of zero.

Print name ______________________ Student number ___________________

Signed ______________________________ Date _________________________