Mr. T. Naughton.

Attempt any THREE questions. Time Allowed: 2 hours.

1. (a) What is the meaning of the Hoare triple \( \{ P \} S \{ R \} \) if it evaluates to true? [3 marks]
   (b) Prove that the following program is correct. [12 marks]
   
   ```
   true
   if odd(x) then
   y := x + 2
   else
   y := x + 1
   fi
   {odd(y) ∧ y > x}
   ```
   (c) Let a function \( f : \mathbb{N} \to \mathbb{N} \) be defined as \( f(x) = x \mod 4 \). Write an expression for the equivalence classes in the partition of \( \mathbb{N} \) induced by the kernel relation of \( f \). [10 marks]

2. (a) Define the language acceptance problem that corresponds to the problem of taking a list of integers and returning the list sorted in ascending order. [3 marks]
   (b) Given the binary relation \( R = \{(1, 2), (2, 3), (3, 4), (4, 5)\} \), construct the relation \( R^3 \). [3 marks]
   (c) Prove that the set of all problems that one may wish a computer to solve is an uncountable set. Use a diagonalisation argument in your proof. [19 marks]
3. (a) Why are language acceptance (language recognition) problems of interest to [5 marks] 
computer scientists?

(b) Calculate the loop invariant $P$ for the following program. [10 marks]

\[
\begin{align*}
\{ x > 0 \land z \geq x \} \\
y := 0; \\
\{ P \} \\
\textbf{while} \ (x + y) < z \ \textbf{do} \\
y := y + 1 \\
\textbf{od} \\
\{ x + y = z \}
\end{align*}
\]

(c) For each of the following sets, state whether the set is finite, countably infinite, [4 marks] 
or uncountable.

i. The set of all real numbers less than 10.

ii. The set of all finite words over a finite alphabet.

iii. The set of all numbers divisible by $\pi$.

iv. The set of all people who are alive or have ever lived.

(d) Let $R$ be the binary relation $R = \{(1, 1), (2, 2), (2, 3), (3, 2), (4, 5)\}$ over [6 marks] 
$\{1, 2, 3, 4, 5\}$. $R$ is not an equivalence relation. Transform $R$, with as little 
modification as possible, so that it becomes an equivalence relation. (Hint: modify $R$ so that it is reflexive, symmetric, and transitive.)

4. (a) Let the set $X$ be defined as $X = \{ x | x \in \mathbb{R} \land 100x \in \mathbb{N} \}$. For example, [8 marks] 
$5.21 \in X$ and $0.99 \in X$. Prove that $X$ is countable.

(b) Prove the correctness or incorrectness of each of the following computer pro- [12 marks] 
grams. You must use the technique based on calculating the most general (or 
weakest) precondition.

i. $\{ x > 0 \} x := x \ast x; x := x \div 2 \{ x^4 = 10 \}$

ii. $\{ \text{true} \} x := 5; x := x + 1 \{ x > 5 \}$

(c) Let $\sim$ be a relation on the natural numbers defined by $x \sim y$ iff $\text{mod}(x, 10) = [5 marks] 
\text{mod}(y, 10)$. Use this relation to partition $\mathbb{N}$. 

SE120 Axioms and Theorems

\[
\begin{array}{ccc}
P & Q & P \rightarrow Q \\
T & T & T \\
T & F & F \\
F & T & T \\
F & F & T \\
\end{array}
\]

Implication truth table:

Assignment axiom (AA): \( \{ Q(t/x) \} x := t \{ Q \} \)

Consequence Rule: \( \frac{P \rightarrow R \text{ and } \{ R \} S \{ Q \}}{\{ P \} S \{ Q \}} \)

Composition Rule: \( \frac{\{ P \} S_1 \{ R \} \text{ and } \{ R \} S_2 \{ Q \}}{\{ P \} S_1; S_2 \{ Q \}} \)

If-Then Rule: \( \frac{\{ P \wedge C \} S \{ Q \} \text{ and } P \wedge \neg C \rightarrow Q}{\{ P \} \text{ if } C \text{ then } S \{ Q \}} \)

If-Then-Else Rule: \( \frac{\{ P \wedge C \} S_1 \{ Q \} \text{ and } \{ P \wedge \neg C \} S_2 \{ Q \}}{\{ P \} \text{ if } C \text{ then } S_1 \text{ else } S_2 \{ Q \}} \)

While Rule: \( \frac{\{ P \wedge C \} S \{ P \}}{\{ P \} \text{ while } C \text{ do } S \{ P \wedge \neg C \}} \)

Selected theorems that can be quoted without proof:
1. The union of any finite number of countable sets is a countable set
2. The cross product of any finite number of countable sets is a countable set
3. The intersection of any finite number of countable sets is a countable set
4. The power set of a finite set is a finite set