1. (a) For each of the following languages, prove that it is regular or prove that it is not regular. [15 marks]
   i. \( L_0 = \{ uv : u, v \in \{0, 1\}^*, u \text{ contains the same number of 1s as } v \} \)
   ii. \( L_1 = \{ uxv : u, v \in \{0, 1\}^*, |u| = |v| \} \)
   iii. \( L_2 = \{ uv : u, v \in \{0, 1\}^*, |u| = |v| \} \)

(b) Let \( A = \{ a : a \in \mathbb{R}, 0 \leq a < 1 \} \). Prove that \( A \) is uncountable. [10 marks]

2. (a) Prove that the set of rational numbers is countable. The set of rational numbers \( \mathbb{Q} \) is defined as \( \mathbb{Q} = \{ \frac{m}{n} : m \in \mathbb{N}, n \in \mathbb{Z} \} \). [5 marks]
   (b) i. Prove that \( L = \{ u x v : u, v \in \{a, b\}^*, |u| = |v|, u \neq v^R \} \) is a context-free language. [8 marks]
   ii. Can a deterministic PDA accept \( L \)? Justify your answer. [2 marks]

(c) Prove that the set of context-free languages is not equal to the set of decidable languages. [5 marks]

(d) How can we use a reduction to prove nonmembership of a class? [5 marks]
Proof. We will use a mapping reduction to prove the reduction \( \square \). Assume that \( \square \) is decidable. The function \( f \) that maps instances of \( \square \) to instances of \( \square \) is performed by TM \( F \) given by the following pseudocode.

\[
F = \text{"On input } \langle \square \rangle \text{:} \\
1. \text{Construct the following } M' \text{ given by the following pseudocode.} \\
   M' = \text{" } \square \text{"} \\
2. \text{Output } \langle \square \rangle \text{"}
\]

Now, \( \langle \square \rangle \) is an element of \( \square \) iff \( \langle \square \rangle \) is an element of \( \square \). So using \( f \) and the assumption that \( \square \) is decidable, we can decide \( \square \). A contradiction. Therefore, \( \square \) is undecidable. (This also means that the complement of \( \square \) is undecidable; the complement of any undecidable language is itself undecidable.)

Figure 1: Proof template for question 4b.

3. (a) Rewrite the function \( f : A \rightarrow 2^B \) as a cross product. [5 marks]

(b) Let \( L = \{ \langle M \rangle : M \text{ is a TM with an input alphabet of } \{a, b\} \text{ and } M \text{ accepts no words} \} \). Prove that \( \overline{L} \), the complement of \( L \), is Turing-recognisable. [10 marks]

(c) Convert the NFA \( N \) into a FA. \( N \) is defined as \( N = \{ Q, \Sigma, \delta, q_0, F \} \), where \( Q = \{99, 01\}, \Sigma = \{a, b\}, q_0 = 99, F = \{99\} \), and \( \delta \) is given by the following table.

\[
\begin{array}{c|ccc}
Q \setminus \Sigma & a & b & e \\
\hline
99 & \{99\} & \{01\} & \{01\} \\
01 & \{01\} & \{01\} & \{01\}
\end{array}
\]

(d) It has been argued that writing programs for NFAs is a waste of time because a NFA could never be built. Comment. [5 marks]

4. (a) You are given that the Hitting Set problem is in \( \mathcal{NP} \). Prove that it is \( \mathcal{NP} \)-complete. [5 marks]

(b) For each of the following languages, prove that it is decidable or prove that it is undecidable. You may use the proof template in Figure 1 if you wish, and if it is necessary. You are given that \( A_{\text{TM}} \) is undecidable. \( A_{\text{TM}} \) is defined as \( A_{\text{TM}} = \{ \langle M, w \rangle : M \text{ is a TM and } M \text{ halts on input } w \} \).

i. \( H_{\text{TM}} = \{ \langle M \rangle : M \text{ is a halting TM} \} \)

ii. \( \text{STATENOENTER}_{\text{TM}} = \{ \langle M, q \rangle : M \text{ is a TM, } q \text{ is a state in } M, \text{ and } M \text{ never goes into state } q \text{ when it is run} \} \)

iii. \( \text{RESPONSE}_{\text{BIN}} = \{ \langle P, f, t \rangle : P \text{ is a program written in machine code for Industrial Automation’s R2 series ELIW processor, } f \text{ is the speed of the processor (number of instruction words per second), } t \text{ is a number of seconds, and when } P \text{ is executed on an R2 processor with speed } f \text{ it halts within } t \text{ seconds} \} \)
Declaration
To be signed by the student and collected by an
invigilator at the beginning of the examination

1. I have searched through my copy of M. Sipser, Introduction to the Theory of Computation, PWS, Boston 1997 (the Sipser book) and it does not contain any extra pages or annotations (except for annotations that correct minor typographical errors).

2. I understand that by failing to notify an invigilator of any annotations or extra pages in my copy of the Sipser book, I will receive a mark of zero in this examination. This does not affect any further disciplinary actions that the University may wish to take.

3. I understand also that directly copying large amounts of material from the Sipser book without substantially tailoring it to the question asked may result in a mark of zero.

Print name ______________________ Student number ________________

Signed ______________________ Date ________________