Discrete Structures 1

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Time allowed: 2 hours

Answer three questions

All questions carry equal marks
1  (a) Given the following propositions P, Q and R:

\[ P: \forall x, \ x+1 > x \]
\[ Q: \forall x, \ x^2 > 0 \]
\[ R: \forall x, \ (x+2)^2 = x^2+4 \]

Determine the truth values for each of the following compound propositions:

\[ \neg \left( (P \lor Q) \land \neg R \right) \]
\[ (P \land \neg Q) \iff R \]

(b) Verify each of the following laws of De Morgan:

\[ \neg (P \lor Q) \equiv \neg P \land \neg Q \]
\[ \neg (P \land Q) \equiv \neg P \lor \neg Q \]

(c) Prove the following propositions. Clearly state the proof strategy used in your solution:

\[ \text{If } n^2 \text{ is odd, then } n \text{ is odd.} \]
\[ \text{If } c \mid a \text{ and } c \mid b, \text{ then } c \mid (am+bn) \text{ for any integers } m \text{ and } n. \]
\[ \text{For any integer } x \text{ and any prime number } p, \text{ if } p \text{ divides } x, \text{ then } p \text{ does not divide } x+1. \]
2 (a) In a survey of 96 students, the following results were recorded: [6 marks]

There are 43 students that live on campus.
There are 38 women in the group.
There are 18 seniors in the group.
There are 11 seniors that live on campus.
There are 10 senior women.
There are 7 women that live on campus.
There are 6 senior women that live on campus.

Provide solutions to the following questions explaining your solution in each case:
• How many women do not live on campus?
• How many senior men are there?
• How many non-senior men do not live on campus?

(b) For each of the following relations, determine whether the relation is a function. If the relation is a function, determine whether the function is injective and/or surjective. For those functions that are bijections determine the inverse function. [6 marks]

\[
\begin{align*}
A &= \{v, w, x, y, z\}, \quad B = \{1, 2, 3, 4, 5\} \\
R &= \{(v, 2), (w, 1), (x, 3), (y, 5)\} \\
\end{align*}
\]

\[
\begin{align*}
A &= \{1, 2, 3, 4, 5\}, \quad B = \{1, 2, 3, 4, 5\} \\
R &= \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\} \\
\end{align*}
\]

(c) Describe Euclid’s Algorithm. Illustrate how you would use Euclid’s Algorithm to calculate the greatest common divisor of 117 and 65. [6 marks]

(d) Consider the function \(f(x, n) = x \mod n\). What is the domain of \(f\)? What is the range of \(f\)? Let \(g(n) = f(209, n)\). Evaluate \(g(g(g(7))))\). [7 marks]
3 (a) Write down each step in the evaluation of $f(13)$ where $f$ has the following recursive definition:

\[
\begin{align*}
f(1) &= 1 \\
f(n) &= f(\text{ceiling}(n/2)) + n.
\end{align*}
\]

(b) Write a recursive definition for the following functions:

- $f: \mathbb{N} \rightarrow \mathbb{N}$ defined by $f(n) = 1 + 5 + 9 + \ldots + (4n+1)$.
- $f: \text{String} \rightarrow \text{String}$ that replaces each occurrence of the letter $a$ by $b$ in a string over the alphabet \{a, b, c\}.

(c) Write an inductive proof to show the following statement is true for all the natural numbers $n \geq 1$:

\[1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{(n(n + 1)(2n+1))}{6}\]

(d) Write down a recursive definition that prints the elements of a list of Integers in reverse order. You may assume the existence of a print procedure that prints a single Integer. Provide an inductive proof to show that the function definition is correct for all input lists.
4. (a) Prove each of the following statements using equivalence laws. Clearly state which law is used at each stage of the proof:

- \((A \rightarrow B) \equiv \neg (A \land \neg B)\)
- \((A \land B \rightarrow C) \rightarrow (A \land C \rightarrow B) \equiv A \land C \rightarrow B\)

(b) Use Quines method to show that the following:

- \((A \rightarrow B) \land (B \rightarrow \neg A) \rightarrow A\) is a contingency
- \((A \lor B) \land (A \rightarrow C) \land (B \rightarrow D) \rightarrow (C \lor D)\) is a tautology

(c) Assume that \(x\) and \(y\) are real numbers and let \(P(x,y)\) denote the predicate \(x + y = 0\). Express each of the following propositions in words and determine their truth value:

- \(\forall x (\exists y (P(x,y)))\)
- \(\exists x (\forall y (P(x,y)))\)

(d) Formalise each of the following English sentences where \(P(x)\) means \(x\) is a person, \(S(x)\) means \(x\) can’t swim, and \(F(x)\) means \(x\) is a fish.

- All fish can swim. John can’t swim. Therefore, John is not a fish.
- Some people can’t swim. All fish can swim. Therefore, there is some person who is not a fish.