Fourier series of a Hard-Sync Sawtooth

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1 Introduction

A *hard-sync sawtooth*, or *hard-sync* for short, is a sawtooth wave whose phase is abruptly reset at regular time intervals.

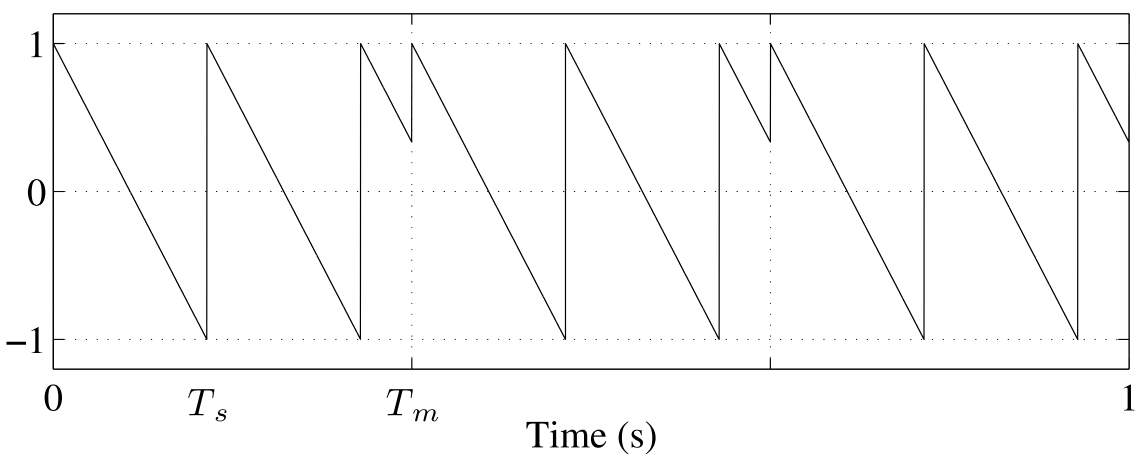

Figure 1: A *Hard-Sync Sawtooth*.

The parameters of such waveform are (amplitude, of course, but also) *master frequency*, $f_\mathrm{m} = 1/T_\mathrm{m}$, and *slave frequency*, $f_\mathrm{s} = 1/T_\mathrm{s}$. Figure 1 gives an example of a hard-sync, with labels. The slave frequency is the frequency of the underlying sawtooth wave. The master frequency is the frequency at which this sawtooth wave is reset in its cycle. The perceived pitch is that of the master frequency, while the timbre is dependent on the slave-to-master frequency ratio.

Hard-Sync Sawtooth waves are used for punchy leads in sound production. They offer an alternative to standard sawtooth waves, with a more "acid" character, and also a timbre that can change dynamically. Traditionally produced with electrical circuits, it is an example of the many waveforms that come under the umbrella of *Virtual Analog Synthesis*, the craft of digitally reproducing waveforms that were the trademark of analog synthesizers. Two essential criteria come into play in the digital reproduction of such waveform : computational efficiency and band-limitedness.

In this document I derive the Fourier series of this waveform, which can be useful to band-limited synthesis. My approach makes extensive use of the equivalences between the Fourier series and the Fourier transform, as well as time-frequency-domain equivalences. This approach has the advantage of yielding great insight in the underlying reality of this waveform's spectrum, and might inspire in the near future methods for efficient synthesis.

2 Summary of the approach

This section describes the sequence of steps that were taken to obtain the hard-sync's Fourier series.

2.1 Express the waveform and Fourier series of a standard sawtooth

Let us consider a standard sawtooth wave of period T_s,

$$s(t) = 1 - \frac{2}{T_\mathrm{s}} \mathrm{mod}(t, T_\mathrm{s}), \tag{1}$$

and its Fourier series,

$$\begin{aligned} S[k] &= \int_{-T_\mathrm{s}/2}^{T_\mathrm{s}/2} s(t) e^{-jk2\pi t/T_\mathrm{s}} dt \\ &= -j\frac{T_\mathrm{s}}{k\pi}. \end{aligned} \tag{2}$$

Figure 2 shows for illustration a sawtooth wave of period T_s, as well as its discrete spectrum. This spectrum is purely imaginary, due to the anti-symmetric nature of the sawtooth wave. Its harmonics are inversely proportional to their index. Finally, notice that the energy of the harmonic of index 0 is nil, as one cycle of the waveform averages to 0.

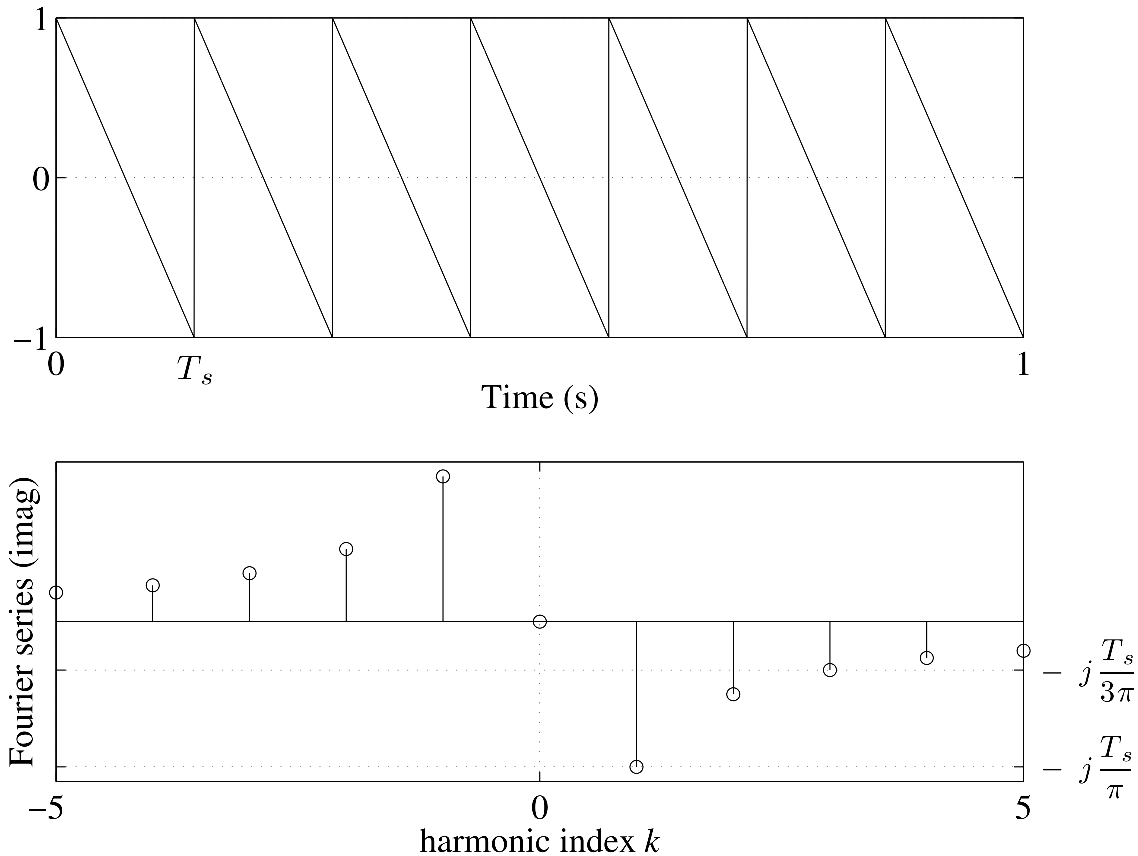

Figure 2: Waveform and Fourier series of a sawtooth of period T_s. The amplitude of the (positive-frequency) first harmonic is $-jT_\mathrm{s}/\pi$, that of the second, $-jT_\mathrm{s}/2\pi$, and so on.

2.2 Infer the Fourier transform of the sawtooth from its Fourier series

We use equality (16) of Appendix B to get the expression of the Fourier transform $S(\omega)$ of the same sawtooth wave :

$$S(\omega) = -j\frac{4\pi}{T_{\mathrm{s}}\omega}\sum_{p}\delta(\omega-\omega_p), \quad \omega \neq 0, \tag{3}$$

where we have made the substitution $p = \omega T_{\mathrm{s}}/2\pi$, and where $\omega_p = p2\pi/T_{\mathrm{s}}$.

2.3 Derive Fourier transform of the rectangular window

Now we derive the Fourier transform $W(\omega)$ of a *rectangular window*

$$w(t) = \begin{cases} 1, & 0 \leq t < T_{\mathrm{m}}, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

The corresponding Fourier transform is easy to derive,

$$\begin{aligned} W(\omega) &= \int_0^{T_{\mathrm{m}}} e^{-j\omega t}dt \\ &= \frac{1}{T_{\mathrm{m}}}\mathrm{sinc}\left(\frac{\omega T_{\mathrm{m}}}{2\pi}\right)e^{j\frac{T_{\mathrm{m}}}{2}\omega}, \end{aligned}$$

where $\mathrm{sinc}(x) \equiv \sin(\pi x)/\pi x$. We show in Figure 3 the waveform and Fourier transform of such rectangular window. To facilitate visualisation, the complex-exponential term of the spectrum was omitted. This term is simply issued from the fact that the window is not centered about zero.

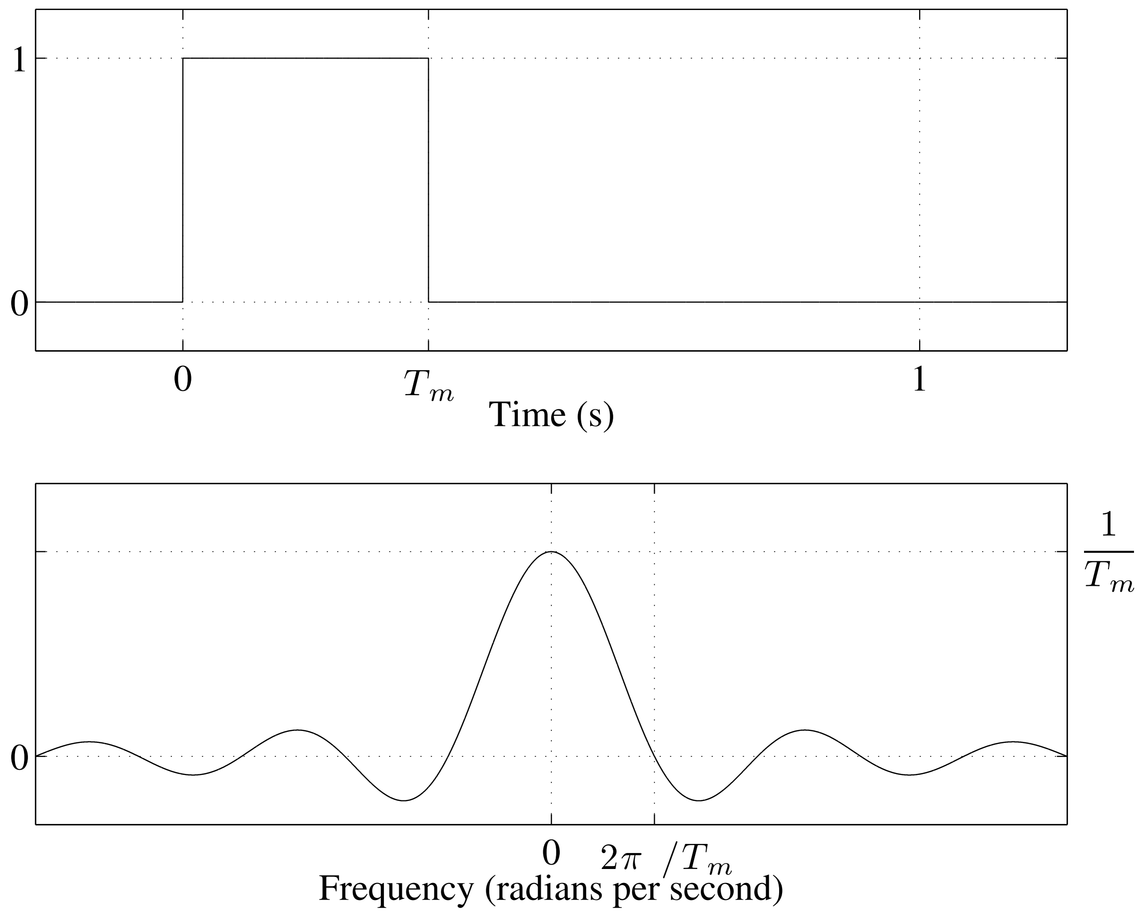

Figure 3: Waveform and Fourier transform of a rectangular window of period $T_{\rm m}$.

2.4 Use the convolution theorem to derive the Fourier transform of the hard-sync

The product of the period-$T_{\rm s}$ sawtooth wave and the length-$T_{\rm m}$ rectangular window yields one cycle of the desired hard-sync sawtooth. As per the *convolution theorem*, the Fourier transform of this product is the convolution of $S(\omega)$ and $W(\omega)$, i.e. $\mathcal{S}_w(\omega) = (S * W)(\omega)$, and this is

$$\mathcal{S}_w(\omega) = -j\frac{2}{T_\mathrm{m}}e^{-j\frac{T_\mathrm{m}}{2}\omega}\sum_{p\neq 0}\frac{1}{p}\mathrm{sinc}\left[(\omega_p - \omega)\frac{T_\mathrm{m}}{2\pi}\right]e^{jp\pi T_\mathrm{m}/T_\mathrm{s}}. \tag{5}$$

The Fourier transform of a windowed hard-sync is given in magnitude in the black line of Figure 4. The coloured lines and other aspects of this figure will shortly be discussed.

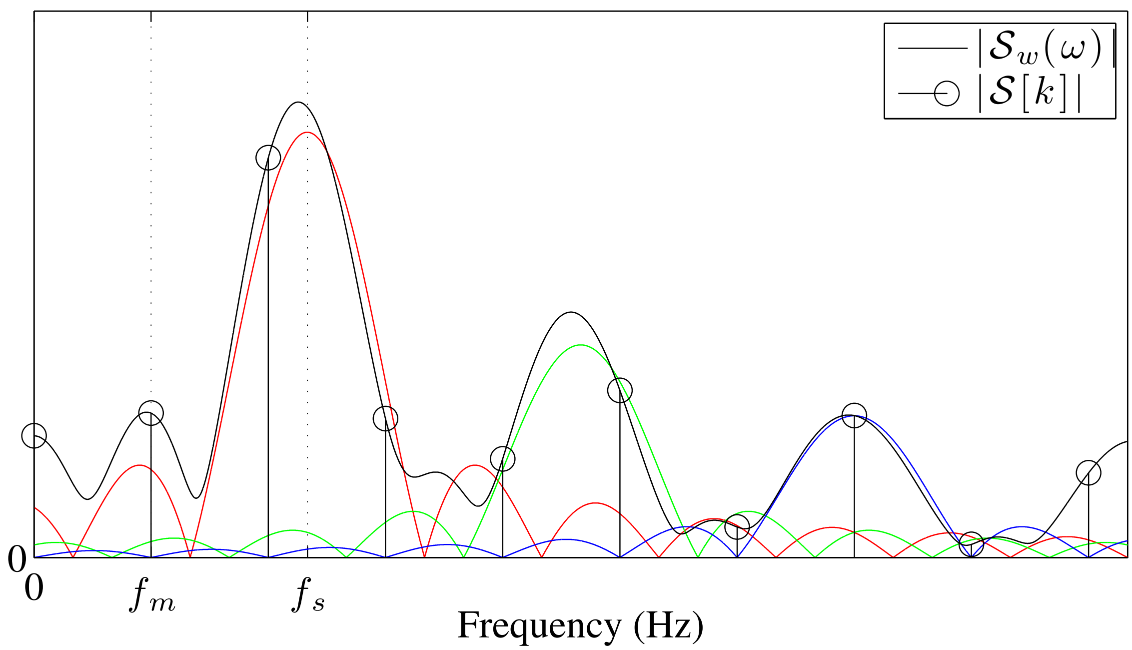

Figure 4: Waveform and Fourier transform of a rectangular window of period T_m.

2.5 Discretise the hard-sync Fourier transform to get its Fourier series

The inverse Fourier transform of $S_w(\omega)$ yields back the windowed hard-sync. The spectrum of the "un-windowed", periodic hard-sync is this same Fourier transform spectrum except sampled every $k2\pi/T_\mathrm{m}$, for all integers k. We substitute our hard-sync's Fourier transform (5) into Equation (17) of Appendix B, and get the hard-sync's Fourier series,

$$\boxed{\mathcal{S}[k] = (-1)^k \frac{1}{j\pi} \sum_{p\neq 0} \frac{1}{p} \mathrm{sinc}\left(p\frac{T_\mathrm{m}}{T_\mathrm{s}} - k\right) e^{jp\pi T_\mathrm{m}/T_\mathrm{s}}}, \qquad (6)$$

which is what we set to do. This series is illustrated in Figure 4 with the stems, and is used to generate the band-limited, Fourier-series-based hard-sync example of Figure 5.

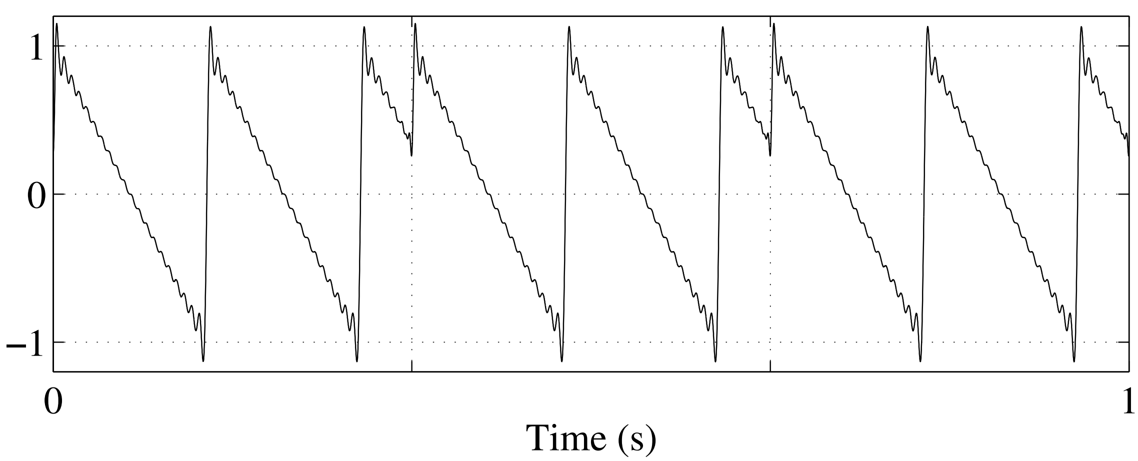

Figure 5: Our inverse-Fourier series hard sync, limiting the number of harmonics to 100, and for each coefficient, the number of cross-leaking sinc functions to 20.

3 Interpretation

You may have noticed that our hard-sync's Fourier series (Equation (6)) is an infinite sum of sinc functions, altogether evaluated at frequencies 0, $2\pi/T_{\mathrm{m}}$, $4\pi/T_{\mathrm{m}}$, and so on (the stems in Figure 4). On the other hand, the Fourier series of the straight sawtooth wave (Equation (2)) does not seem to feature such infinite sum. Well, actually it does, but what happens then is, each of the sinc function is centered on a harmonic of the sawtooth and crosses the zero axis exactly where the other harmonics stand. This way, each sinc function contributes to the harmonic it is centered on only, and is invisible to the others. In such case, we might say that the sawtooth and window spectra are "tuned". In Figure 6, for instance, the period of the window was aligned on that of the sawtooth. Now, the red-coloured sinc only contributes to the first harmonic, and is 0 for all other harmonics – including harmonic 0. Likewise, the green-coloured sinc only contributes to the second harmonic, and the blue-coloured sinc, to the third harmonic.

On the one hand, the width of the window is going to determine how narrow the main lobe of the sinc function is and how often it crosses the zero-axis, i.e. every $2\pi/T_{\mathrm{m}}$ radians per second. On the other hand, it is the frequency of the underlying sawtooth wave which is going determine upon which frequencies the sinc functions are centered, i.e. every multiple of $2\pi/T_{\mathrm{s}}$

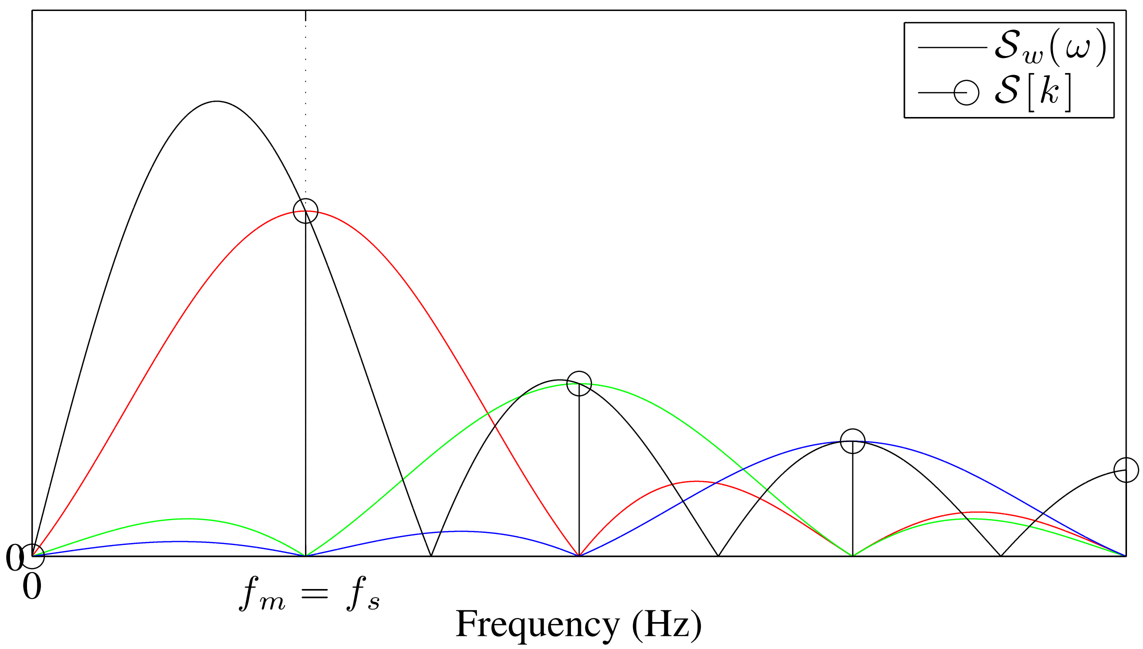

Figure 6: Waveform and Fourier transform of a rectangular window of period T_{m}.

radians per second. What is observed in Figure 4 is the result of augmenting the width of the window – and hence narrowing its main lobe and zero-crossing interval – while preserving the frequency of the sawtooth, and hence the frequency alignment of the window's sincs. The side-effect of such change is the cross-leakage of all these components into one another. Even the zeroth harmonic is now non-zero.[1] This makes the synthesis of the hard-sync's Fourier series more delicate. Here, I am just going to throw a few suggestions that I can find on top of my head as to what approach might be taken.

First, it is not possible to account for an infinite sum ; it has to be truncated. In this regard, one should consider the inversely proportional decay of the sinc function, which reduces the influence of their leakage the further apart the harmonics are and the narrower their main lobe is. Will therefore be subject to less cross-leakage hard-sync waveforms where the period of the sawtooth is much smaller than the length of the window. Anyway, the number of sinc functions taken into account for the computation of each Fourier coefficient may be determined on this basis.

Second, one may use a table read-out of the sinc function. A sinc table may be computed at initialisation time, and its values

[1]This can alternatively be explained by the fact that one cycle of the hard-sync waveform does not average to zero anymore, and that the energy of the zero-frequency component is determined by this average. Isn't it beautiful how consistent with this simplistic interpretation the sinc-reading is, all the while far more abstract?

read-out with linear interpolation, rather than synthesizing a number of computationally expensive sync values every frame. After that, a Phase-Vocoder approach might be taken, or a more traditional additive synthesis approach. Both allow for the dynamic change of the parameters of this musically interesting waveform. In Figure 7, additive synthesis is used to create two examples.

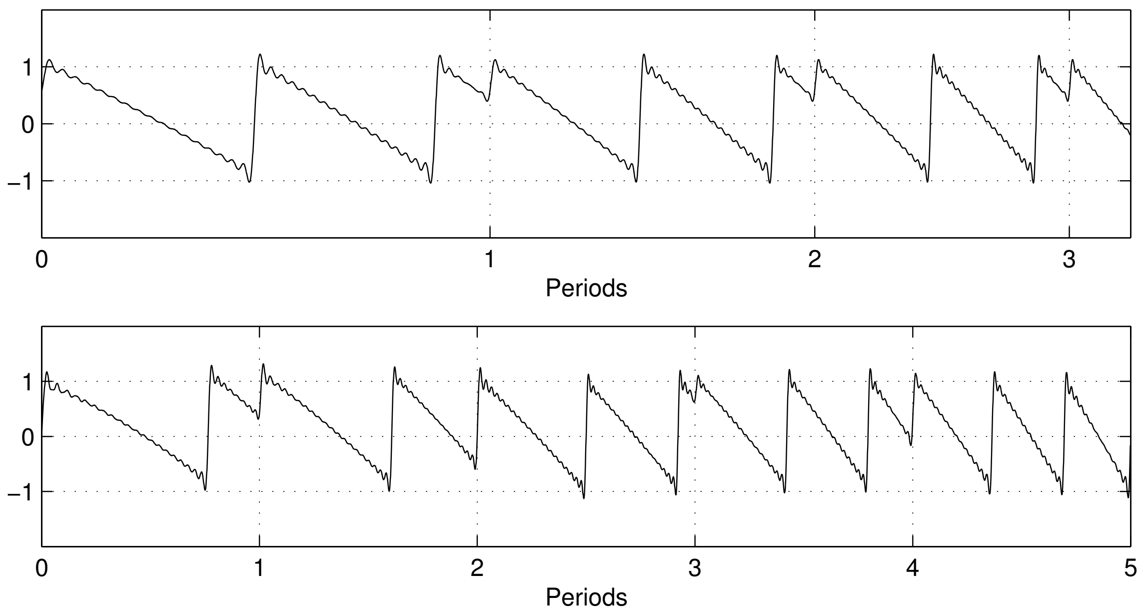

Figure 7: A Fourier-based hard-sync with dynamic parameters. Top : constant slate-to-master frequency ratio, increasing master frequency ; Bottom : constant master frequency, increasing slave-to-master frequency ratio.

Topmost is the example of a hard-sync whose slave-to-master frequency ratio stays constant (7/3), but it's master (fundamental) frequency increases. One can see the waveform pattern remain the same, except at an increasing rate. The equivalent audio example, preserving this 7/3 ratio but for a master frequency gliding from 40 to 2,000Hz, can be heard at the url here.[2]

Bottommost, the master frequency is kept constant, but the slave-to-master frequency is gradually increased. At audio rate, this creates the sensation of a waveform of constant pitch, but with a dynamic timbre. An audio file is available here,[3] where the master frequency is kept at 110Hz, and the slave-to-master frequency ratio gradually increases from 1 to 3.

4 Conclusion

The main contribution of this document is the formulation of the Fourier series of a hard-sync sawtooth wave, found in Equation (6). To obtain this Fourier series, we have looked at it as the discretisation of the convolution of the Fourier transform of a sawtooth wave of period T_s and the Fourier transform of a rectangular window of length T_m. This Fourier series features an infinite sum of sinc functions. More effort should now be put in finding a way to approximate this spectrum efficiently. This Fourier series can be used as a starting point for the band-limited synthesis of the hard-sync waveform.

[2] http://www.cs.nuim.ie/~matthewh/hardSyncChirp.mp3

[3] http://www.cs.nuim.ie/~matthewh/hardSyncTimbre.mp3

A Fourier transform and Fourier series : definition

A.1 Fourier transform

The Fourier transform of a signal $s(t)$ is

$$S(\omega) = \int s(t)e^{-j\omega t}dt, \tag{7}$$

where $\int dt$ is a shorthand for $\int_{-\infty}^{\infty} dt$.

The inverse Fourier transform, in turn, is

$$s(t) = \frac{1}{2\pi} \int S(\omega)e^{j\omega t}dt. \tag{8}$$

A.2 Fourier series

In this document we use the two-sided, complex Fourier series :

$$S[k] = \int_{-T/2}^{T/2} s(t)e^{-jk2\pi t/T}, \tag{9}$$

whose inverse is

$$s(t) = \frac{1}{T} \sum_k S[k]e^{jk2\pi t/T}, \tag{10}$$

where $\sum_k$ is a shorthand for $\sum_{k=-\infty}^{\infty}$.

B Fourier transform / Fourier series equivalence

We aim at establishing the equivalence between the Fourier series and transform of a unique waveform, periodic in T. Both its inverse Fourier series and transform should produce the same result,

$$\begin{aligned} s(t) &= \frac{1}{T}\sum_k S[k]e^{jk2\pi t/T} \\ &= \frac{1}{2\pi}\int S(\omega)e^{j\omega t}d\omega. \end{aligned} \tag{11}$$

Where (11) is concerned, the condition for $s(t)$ to be periodic in T is that $S(\omega)$ is a train of complex impulses, of frequency $2\pi/T$,

$$S(\omega) = \sum_k \tilde{A}_k \delta(\omega - k2\pi/T), \tag{12}$$

where $\tilde{A}$ is an arbitrary complex value. By substitution, the inverse Fourier transform of $S(\omega)$ reduces to

$$s(t) = \frac{1}{2\pi}\sum_k \tilde{A}_k e^{jk2\pi t/T}. \tag{13}$$

We can now re-state equality (11) as

$$\frac{1}{T}\sum_k S[k]e^{jk2\pi t/T} = \frac{1}{2\pi}\sum_k \tilde{A}_k e^{jk2\pi t/T}, \tag{14}$$

which holds for

$$\tilde{A}_k = \frac{2\pi}{T} S[k]. \tag{15}$$

Now (12) becomes

$$S(\omega) = \frac{2\pi}{T} \sum_k S[k] \delta(\omega - k 2\pi/T), \tag{16}$$

which is equivalent to stating that

$$S[k] = \frac{T}{2\pi} S(\omega_k), \quad \omega_k = k 2\pi/T. \tag{17}$$

Equation (16) tells us that the Fourier transform of a signal periodic in T is a frequency-domain complex pulse train, whose values are the signal's Fourier series', scaled by $2\pi/T$. Conversely, Equation (17) tells us that the Fourier series of a signal taken over a length-T interval can be obtained by the $T/2\pi$-scaled readout of its Fourier transform at frequency intervals $k2\pi/T$.