



NUI MAYNOOTH

Ollscoil na hÉireann Má Nuad

OLLSCOIL NA hÉIREANN MÁ NUAD

**THE NATIONAL UNIVERSITY OF IRELAND
MAYNOOTH**

JANUARY 2014 EXAMINATION

CS605

**The Mathematics and Theory
of Computer Science**

Dr. L. Rapanotti, Dr. A. Winstanley, Mr. T. Naughton

Time allowed: 3 hours

Answer **three** questions

All questions carry equal marks

Additional material allowed:

Copies of M. Sipser, *Introduction to the Theory of Computation* (any editions), without annotations or extra pages. Please see declaration page at back.

[25 marks]

- 1 (a) The language $\{wxw : w \in \{0, 1\}^*\}$ is Turing-recognisable.
- i. Outline a pseudocode algorithm (using English language, not a Turing machine table of behaviour) for a single-tape Turing machine to recognise this language. [3 marks]
- ii. In terms of the length of w , calculate the exact number of timesteps your Turing machine would require to accept a word in the worst case (some flexibility will be allowed in calculating the exact number of timesteps due to the inherent imprecise nature of your pseudocode algorithm). [2 marks]
- iii. Give your answer to part ii for your Turing machine in asymptotic notation. [2 marks]
- (b) Prove that the set of all words over a finite alphabet is countable. [4 marks]
- (c) Prove that the set of all languages over a finite alphabet is uncountable. [11 marks]
- (d) State whether each of the following is true or false. Justify your answer. [3 marks]
- i. $\emptyset \in \emptyset$
- ii. $\emptyset = 2^\emptyset$
- iii. $\{a, b\} \subseteq 2^{\{a, b, \{a, b\}\}}$

[25 marks]

- 2 (a) Consider the following sextuple of language classes: (regular, context-free, Turing-recognisable, decidable, \mathcal{P} , \mathcal{NP}).

Each language can be associated with a binary sextuple where symbol 1 denotes membership and 0 denotes nonmembership of the class in question. For example, if a language was in the first two classes and not in any of the others, it would be associated with the binary sextuple (1,1,0,0,0,0). State the binary sextuple associated with each of the following languages.

i. The language $L = \{w : w \in \{a,b\}^*, |w| \text{ is odd, } w \text{ has exactly one } b \text{ positioned exactly in the centre of the word}\}$. [2 marks]

ii. The language of pairs (containing a chess board configuration plus a legal move for the white player) for which that is the best next move for white. [3 marks]

iii. The language of Turing machines that go into a state 01 when executed. [3 marks]

Finally, state two (2) different binary sextuples that cannot exist, and indicate where the contradiction occurs. [2 marks]

- (b) Let $L = \{\langle M \rangle : M \text{ is a Turing machine with an input alphabet of } \{a,b\} \text{ and } M \text{ accepts at most one word, i.e. } M \text{ either accepts no words or accepts exactly one word}\}$. Prove that the complement of L is Turing-recognisable. [12 marks]

- (c) How can we use a reduction to prove nonmembership of a class? [3 marks]

- [25 marks]**
- 3 (a) What are the steps required to prove that a language is \mathcal{NP} -complete? [4 marks]
- (b) You want to find an algorithmic solution for problem A . You know that A is in \mathcal{NP} .
- Should you look for an efficient algorithm for A ? Explain. [3 marks]
- Continuing this question, given an arbitrary problem in \mathcal{NP} , what is the likelihood that it has provably low computational complexity? [1 mark]
- (c) For each of the following languages, prove that it is regular or prove that it is not regular. [10 marks]
- i. $\{w : w \in \{a, b\}^*, w \text{ contains a different number of substrings } ba \text{ as } ab\}$ (e.g. the string ab is in this language but the string aba is not in this language)
- ii. $\{w\#ww : w \in \{a, b\}^*\}$
- iii. $\{uv : u, v \in \{a, b\}^*, |u| = c|v|, c \text{ is an integer and } c \geq 0\}$
- (d) Outline in detail how one could prove that the set of regular languages is a proper subset of the set of the context-free languages. You must explain each sub-proof required and give a detailed plan for how you would prove each sub-proof. The only theorems you may use (if you wish to) are those you have proved from part (c) of this question and the following [7 marks]
- a language is regular iff it is accepted by a finite automaton
 - a language is context-free iff it is accepted by a pushdown automaton.

[25 marks]

- 4 (a) What does a reduction $A \leq B$ between two problems A and B establish about the relative computability of A and B? What does a polynomial reduction establish about the relative computational complexity of A and B? [4 marks]
- (b) Let the language $NLESS_{C++}$ be defined as $NLESS_{C++} = \{ \langle A, b, c \rangle : A \text{ is a C++ program, and } b \text{ and } c \text{ are integer variables declared in } A, \text{ and throughout the execution of } A, b \text{ never has a value less than the value of } c \}.$ You are given that $HALT_{C++}$ is undecidable. $HALT_{C++}$ is defined as $HALT_{C++} = \{ \langle B, w \rangle : B \text{ is a C++ function, and } B \text{ halts on its string input } w \}.$ Prove that $NLESS_{C++}$ is undecidable. You may answer this question by assigning a name, mathematical construct, or piece of pseudocode to each of the numbered blanks in the proof template in Figure 1 on page 4. Where blanks have the same number, this denotes that their contents will be the same. Alternatively, you can choose to ignore the template and construct your own proof from scratch. [13 marks]
- (c) Prove that $NLESS_{C++}$ is Turing-recognisable or prove that it is not Turing-recognisable. [4 marks]
- (d) Give a definition of the language $LESS_{C++}$ (the complement of $NLESS_{C++}$). Prove that $LESS_{C++}$ is Turing-recognisable or prove that it is not Turing-recognisable. [4 marks]

Proof. We will use a mapping reduction to prove the reduction 1. Assume that 2 is decidable. The function f that maps instances of 3 to instances of 4 is performed by TM F given by the following pseudocode.

$F =$ “On input $\langle \underline{5} \rangle$:

1. Construct the following M' given by the following pseudocode.

$M' =$ “ 6 ”

2. Output $\langle \underline{7} \rangle$ ”

Now, $\langle \underline{7} \rangle$ is an element of 8 iff $\langle \underline{5} \rangle$ is an element of 9. So using f and the assumption that 2 is decidable, we can decide 10. A contradiction. Therefore, 2 is undecidable. (This also means that the complement of 2 is undecidable; the complement of any undecidable language is itself undecidable.)

Figure 1. Proof template that can be used if you wish.



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Declaration

To be signed by the student and collected by an
invigilator at the beginning of the examination

1. I have searched through my copies of M. Sipser, *Introduction to the Theory of Computation*, any editions, (the Sipser book) and it does not contain any extra pages or annotations (except for annotations that correct minor typographical errors).
2. I understand that by failing to notify an invigilator of any annotations or extra pages in my copies of the Sipser book, I will receive a mark of zero in this examination. This does not affect any further disciplinary actions that the University may wish to take.
3. I understand also that directly copying large amounts of material from the Sipser books without substantially tailoring it to the question asked may result in a mark of zero.

Print name _____ Student number _____

Signed _____ Date _____