



NUI MAYNOOTH

Ollscoil na hÉireann Má Nuad

OLLSCOIL NA hÉIREANN MÁ NUAD

THE NATIONAL UNIVERSITY OF IRELAND MAYNOOTH

JANUARY 2012 EXAMINATION

CS605

The Mathematics and Theory of Computer Science

Dr. L. Rapanotti, Dr. A. Winstanley, Mr. T. Naughton

Time allowed: 3 hours

Answer **three** questions

All questions carry equal marks

Additional material allowed:

One copy of M. Sipser, *Introduction to the Theory of Computation* (PWS, Boston, 1997), containing no annotations or extra pages AND/OR one copy of M. Sipser, *Introduction to the Theory of Computation – Second Edition* (Thomson, Boston, 2005), containing no annotations or extra pages. Please see declaration page at back.

- 1** (a) State whether each of the following is true or false. **[25 marks]**
[6 marks]
- i. $\emptyset \in \emptyset$
 - ii. $\emptyset = 2^\emptyset$
 - iii. $\{a, b\} \subseteq 2^{\{a, b, \{a, b\}\}}$
- (b) Let $\Sigma = \{a, b, c\}$ and let $L = \{w : w \in \Sigma^*\}$. [5 marks]
- Write down the first five elements in the lexicographical ordering of L , where Σ has the usual alphabetical ordering (a, b, c) .
- (c) Let \mathbb{R} be the set of real numbers. Let \mathbb{Z} be the set of integers. Let \mathbb{N} be the set of natural numbers. State whether each of the following sets is countable or not. [8 marks]
- i. $2^{\mathbb{R}}$
 - ii. $2^{\mathbb{Z}}$
 - iii. $2^{\mathbb{N}}$
 - iv. $2^{\mathbb{R} \cap \mathbb{Z}}$
- (d) Can you enumerate the set of all words over a finite alphabet? [6 marks]
Prove your answer.

- 2** **[25 marks]**
- A software company wants to improve the efficiency of function call parameter passing in its software products. In the existing code, there were many instances of objects being passed by value to functions that did not change these objects under any circumstances. In order to avoid the overhead of implicitly copying the objects when such functions are called it would be preferable to pass the objects by reference. The company is writing a software routine that takes a description of a function $f()$ as input and determines whether its arguments (currently passed by value) would be updated during the execution of $f()$ or not. If an argument would not be updated then it could be safely passed by reference without changing the semantics of the program. (Assume that functions only refer to variables passed as arguments or declared within the function body – i.e. there are no global variables to contend with).
- (a) Consider a restricted version of the problem where $f()$ contains only one argument A and there are no other function calls within the body of $f()$. Prove the undecidability of this problem. You are given that $\text{HALTS} = \{\langle M \rangle : M \text{ is a Turing machine that halts}\}$ is undecidable. [19 marks]

[Continued overleaf]

- (b) Consider the more general case where $f()$ contains an arbitrary number of arguments (A, B, C, \dots and so on) and $f()$ can contain calls to itself or other functions. Prove the decidability or undecidability of this problem. You may refer to your solution to part (a) in your proof. [6 marks]

- 3 (a) Consider the following sextuple of language classes: (regular, context-free, Turing-recognisable, decidable, \mathcal{P} , \mathcal{NP}). [25 marks]

Each language can be associated with a binary sextuple where symbol 1 denotes membership and 0 denotes nonmembership of the class in question. For example, if a language was in the first two classes and not in any of the others, it would be associated with the binary sextuple (1,1,0,0,0,0). State the binary sextuple associated with each of the following languages.

- i. The language $L = \{a, aa, ab\}$. [1 mark]
 - ii. The language $L = \{w : w \in \{a, b\}^*, |w| \text{ is odd, } w \text{ has exactly one } b \text{ positioned exactly in the centre of the word}\}$. [3 marks]
 - iii. The language $L = \{w : w \in \{a, b\}^*, |w| \text{ is odd, } w \text{ has exactly one } b\}$. [3 marks]
 - iv. The language of chess board configurations for which white can win. [3 marks]
 - v. The language of Turing machines that contain a state 01. [3 marks]
 - vi. The language of Turing machines that go into a state 01. [3 marks]
 - vii. The language of set systems that can be hit by a set of cardinality 12 (where 'hit' is a technical term defined in the well-known Hitting Set problem). [3 marks]
 - viii. The language of graphs that have a tour that visits each vertex at least once. [3 marks]
- (b) State three binary sextuples that cannot exist, and indicate where the contradiction occurs in each case. [3 marks]

- 4 (a) What is a polynomially-bounded (or polynomial-time) reduction? [25 marks]
[5 marks]
Illustrate with an example why these are of interest in computational complexity theory.
- (b) You want to find an algorithmic solution for problem A . You [4 marks]
know that A is in \mathcal{NP} .
Should you look for an efficient algorithm for A ? Explain.
- (c) Is the problem of writing out the factorial of a number in unary [4 marks]
 \mathcal{NP} -complete or \mathcal{NP} -hard (e.g. $n! = 111111$ for $n = 3$)? Explain.
- (d) A nationwide clothing shop chain, Smyth Bros., decides to lower [12 marks]
the price of selected pairs or trousers. Unfortunately, while the larger Smyth Bros. outlets sell each of the m brands, some of the smaller outlets sell but a limited range. Smyth Bros. wants to discount no more than k possible brands such that each of its n outlets stocks at least one discounted item. “Surely a computer this powerful can figure it out,” one of the board members was heard to say pointing to a top-of-the-range PC in the head office. Prove that this problem is \mathcal{NP} -complete.
- You are given only that the problem SATISFIABILITY (also known as SAT) is \mathcal{NP} -complete.

Proof. We will use a mapping reduction to prove the reduction 1. Assume that 2 is decidable. The function f that maps instances of 3 to instances of 4 is performed by TM F given by the following pseudocode.

$F =$ “On input $\langle \underline{5} \rangle$:

1. Construct the following M' given by the following pseudocode.

$M' =$ “ 6 ”

2. Output $\langle \underline{7} \rangle$ ”

Now, $\langle \underline{7} \rangle$ is an element of 8 iff $\langle \underline{5} \rangle$ is an element of 9. So using f and the assumption that 2 is decidable, we can decide 10. A contradiction. Therefore, 2 is undecidable. (This also means that the complement of 2 is undecidable; the complement of any undecidable language is itself undecidable.)

Figure 1. Proof template that can be used if you wish.



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Declaration

To be signed by the student and collected by an
invigilator at the beginning of the examination

1. I have searched through my copies of M. Sipser, *Introduction to the Theory of Computation*, first and second editions, (the Sipser book) and it does not contain any extra pages or annotations (except for annotations that correct minor typographical errors).
2. I understand that by failing to notify an invigilator of any annotations or extra pages in my copies of the Sipser book, I will receive a mark of zero in this examination. This does not affect any further disciplinary actions that the University may wish to take.
3. I understand also that directly copying large amounts of material from the Sipser books without substantially tailoring it to the question asked may result in a mark of zero.

Print name _____ Student number _____

Signed _____ Date _____