NUI MAYNOOTH

Ollscoil na hÉireann Má Nuad

OLLSCOIL NA hÉIREANN MÁ NUAD

THE NATIONAL UNIVERSITY OF IRELAND MAYNOOTH

JANUARY 2011 EXAMINATION

CS605

The Mathematics and Theory of Computer Science

Dr. L. Rapanotti, Dr. A. Winstanley, Mr. T. Naughton

Time allowed: 3 hours

Answer ***three*** questions

All questions carry equal marks

Additional material allowed:
One copy of M. Sipser, *Introduction to the Theory of Computation* (PWS, Boston, 1997), containing no annotations or extra pages AND/OR one copy of M. Sipser, *Introduction to the Theory of Computation – Second Edition* (Thomson, Boston, 2005), containing no annotations or extra pages. Please see declaration page at back.

[25 marks]

1 (a) A student claims that the intersection of two regular languages is a regular language, and attempts to prove it by picking two specific regular languages and showing that their intersection is regular. [10 marks]

i. What is insufficient with this student's proof?

ii. Prove that this claim (closure under intersection) is true for all regular languages.

(b) Let L = {<M> : M is a TM with an input alphabet of {a,b} and M accepts at most one word, i.e. M either accepts no words or accepts exactly one word}. Prove that the complement of L is Turing-recognisable. [12 marks]

(c) How can we use a reduction to prove nonmembership of a class? [3 marks]

[25 marks]

2 (a) For each of the following languages, prove that it is regular or prove that it is not regular. Note, the marks are not divided equally between each part (more marks will be given for proving a language is not regular than for proving a language is regular). [15 marks]

i. $\{w : w \in \{a, b\}^*$, w is the empty word or contains substring aab$\}$

ii. $\{ww : w \in \{a, b\}^*\}$

iii. $\{uv : u,v \in \{a, b\}^*$, u is not equal to v$\}$

iv. $\{w : w \in \{a,b\}^*$, w contains an equal number of occurrences of substring ab and substring ba, e.g. the word aba contains one occurrence of the substring ab and one occurrence of the substring ba$\}$

(b) Prove that $L = \{uvw : u,v,w \in \{a,b\}^*, |u| = |w|, u = w^R\}$ is a context-free language. Can a deterministic pushdown automaton recognise this language? [10 marks]

[25 marks]

3 (a) The language $\{a^p b^p : p > 0\}$ is Turing-recognisable.

i. Outline a pseudocode algorithm (using English language, not a Turing machine table of behaviour) for a single-tape Turing machine to recognise this language. [3 marks]

ii. For an input size of n, calculate the exact number of timesteps your Turing machine would require to accept a word (some flexibility will be allowed in calculating the exact number of timesteps due to the inherent imprecise nature of your pseudocode algorithm) [2 marks]

iii. Give your answer to part ii for your Turing machine in asymptotic notation. [2 marks]

(b) Prove that the set of all words over a finite alphabet is countable. [4 marks]

(c) Prove that the set of all languages over a finite alphabet is uncountable. [10 marks]

(d) What does a reduction $A \leq B$ between two problems A and B establish about the relative computability of A and B? What does a polynomial reduction establish about the relative computational complexity of A and B? [4 marks]

[25 marks]

4 (a) Why do we use languages to study the power of computing devices? [5 marks]

(b) For each of the following languages, prove that it is decidable or prove that it is undecidable. Note, the marks are not divided equally between each part (more marks will be given for proving a language is undecidable than for proving a language is decidable). If appropriate, you may answer this question by assigning a name, mathematical construct, or piece of pseudocode to each of the numbered blanks in the proof template in Figure 1 on page 4. Where blanks have the same number, this denotes their contents will be the same. Alternatively, if appropriate, you can choose to ignore the template and construct your own proof from scratch. You are given that A_{TM} is undecidable. A_{TM} is defined as A_{TM} = {<M, w> : M is a TM and M halts on input w}. [20 marks]

i. $LT5_{TM}$ = {<M> : M is Turing machine that halts if given a word of length less than 5}

ii. $NEVERACCESS_{C++}$ = {<M, x> : M is a C++ program, x is an integer variable declared in M, and M never reads from or writes to x when it is run}

iii. $VAREQUAL_J$ = {<M, x, y> : M is a Java program, x and y are integer variables declared in M, and M contains a line of code assigning the value of x to y}

Proof. We will use a mapping reduction to prove the reduction __1__. Assume that __2__ is decidable. The function f that maps instances of __3__ to instances of __4__ is performed by TM F given by the following pseudocode.

$F =$ "On input $\langle$__5__$\rangle$:

1. Construct the following M' given by the following pseudocode.

   $M' =$ "__6__"

2. Output $\langle$__7__$\rangle$"

Now, $\langle$__7__$\rangle$ is an element of __8__ iff $\langle$__5__$\rangle$ is an element of __9__. So using f and the assumption that __2__ is decidable, we can decide __10__. A contradiction. Therefore, __2__ is undecidable. (This also means that the complement of __2__ is undecidable; the complement of any undecidable language is itself undecidable.)

Figure 1. Proof template that may be appropriate for question 4 on page 3.

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Declaration

To be signed by the student and collected by an invigilator at the beginning of the examination

1. I have searched through my copies of M. Sipser, *Introduction to the Theory of Computation*, first and second editions, (the Sipser book) and it does not contain any extra pages or annotations (except for annotations that correct minor typographical errors).

2. I understand that by failing to notify an invigilator of any annotations or extra pages in my copies of the Sipser book, I will receive a mark of zero in this examination. This does not affect any further disciplinary actions that the University may wish to take.

3. I understand also that directly copying large amounts of material from the Sipser books without substantially tailoring it to the question asked may result in a mark of zero.

Print name ________________________ Student number ______________________

Signed ________________________________ Date ________________________