



NUI MAYNOOTH

Ollscoil na hÉireann Má Nuad



JANUARY 2009 EXAMINATION

CS605

The Mathematics and Theory of Computer Science

Dr. P. Sawyer, Dr. A. Winstanley, Mr. T. Naughton

Time allowed: 3 hours

Answer **three** questions

All questions carry equal marks

Additional material allowed:

One copy of M. Sipser, *Introduction to the Theory of Computation* (PWS, Boston, 1997), containing no annotations or extra pages AND/OR one copy of M. Sipser, *Introduction to the Theory of Computation – Second Edition* (Thomson, Boston, 2005), containing no annotations or extra pages. Please see declaration page at back.

- 1 (a) State whether each of the following is true or false. The symbol \emptyset denotes the empty set. [6 marks]
- i. $\emptyset \in \emptyset$
 - ii. $\emptyset \in 2^\emptyset$
 - iii. $\{a, b\} \subseteq 2^{\{a, b, \{a, b\}\}}$
- (b) Let $\Sigma = \{a, b, c\}$ and let $L = \{w : w \in \Sigma^*\}$. Write down the first five elements in the lexicographical ordering of L , where Σ has the usual alphabetical ordering (a, b, c). [5 marks]
- (c) Let \mathbb{R} be the set of real numbers. Let \mathbb{Z} be the set of integers. Let \mathbb{N} be the set of natural numbers. State whether each of the following sets is countable or not. The symbol \cap denotes intersection. [8 marks]
- i. $2^{\mathbb{R}}$
 - ii. $2^{\mathbb{Z}}$
 - iii. $2^{\mathbb{N}}$
 - iv. $2^{\mathbb{R} \cap \mathbb{Z}}$
- (d) Can you enumerate the set of all words over a finite alphabet? Prove your answer. [6 marks]
- 2 (a) A software company wants to improve the efficiency of function call parameter passing in its software products. In the existing code, there were many instances of objects being passed by value to functions that did not change these objects under any circumstances. In order to avoid the overhead of implicitly copying the objects when such functions are called it would be preferable to pass the objects by reference. The company wishes to write a routine to solve the following problem: given a description of a function $f()$ as input, how can one determine whether its arguments (currently passed by value) would be updated during the execution of $f()$ or not. If an argument would not be updated then it could be safely passed by reference without changing the semantics of the program. (Assume that functions only refer to variables passed as arguments or declared within the function body – i.e. there are no global variables to contend with). [20 marks]
- i. Consider a restricted version of the problem where $f()$ contains only one argument A and there are no other function calls within the body of $f()$. Prove the decidability or undecidability of this problem.
 - ii. Consider the more general case where $f()$ contains an arbitrary number of arguments (A, B, C, \dots and so on) and $f()$ can contain calls to itself or other functions. Prove the decidability or undecidability of this problem.
- (b) Why do we use languages to study the power of computing devices? [5 marks]

- 3 (a) What are the steps required to prove that a language is NP-complete? [4 marks]
- (b) Let α be an arbitrary number that is the product of two prime numbers. Prove that the problem of factoring α is in NP. [4 marks]
- (c) For each of the following languages, prove that it is regular or prove that it is not regular. [10 marks]
- i. $\{w : w \in \{a, b\}^*, w \text{ is the empty word or contains substring } aab\}$
 - ii. $\{ww : w \in \{a, b\}^*\}$
 - iii. $\{uv : u, v \in \{a, b\}^*, u \text{ is not equal to } v\}$
- (d) Outline in detail how one could prove that the set of regular languages is a proper subset of the set of the context-free languages. You must explain each sub-proof required and give a detailed plan for how you would prove each sub-proof. The only theorems you may use (if you wish to) are those you have proved from part (c) of this question and the following [7 marks]
- a language is regular iff it is accepted by a finite automaton
 - a language is context-free iff it is accepted by a pushdown automaton.
- 4 (a) What does a reduction $A \leq B$ between two problems A and B establish about the relative computability of A and B? What does a polynomial reduction establish about the relative computational complexity of A and B? [4 marks]
- (b) Let the language NTWOTIMES_{C++} be defined as $\text{NTWOTIMES}_{C++} = \{ \langle A, b \rangle : A \text{ is a C++ program, and } b \text{ is an integer variable declared in } A, \text{ and when } A \text{ is run } b \text{ is decremented at most once during the execution of } A \}$. You are given that HALT_{C++} is undecidable. HALT_{C++} is defined as $\text{HALT}_{C++} = \{ \langle B, w \rangle : B \text{ is a C++ function, and } B \text{ halts on its string input } w \}$. Prove that NTWOTIMES_{C++} is undecidable. You may answer this question by assigning a name, mathematical construct, or piece of pseudocode to each of the numbered blanks in the proof template in Figure 1 on page 4. Where blanks have the same number, this denotes their contents will be the same. Alternatively, you can choose to ignore the template and construct your own proof from scratch. [13 marks]
- (c) Prove that NTWOTIMES_{C++} is Turing-recognisable or prove that it is not Turing-recognisable. [4 marks]
- (d) Give a definition of the language TWOTIMES_{C++} (the complement of NTWOTIMES_{C++}). Prove that TWOTIMES_{C++} is Turing-recognisable or prove that it is not Turing-recognisable. [4 marks]

Proof. We will use a mapping reduction to prove the reduction 1. Assume that 2 is decidable. The function f that maps instances of 3 to instances of 4 is performed by TM F given by the following pseudocode.

$F =$ “On input $\langle \underline{5} \rangle$:

1. Construct the following M' given by the following pseudocode.

$M' =$ “6”

2. Output $\langle \underline{7} \rangle$ ”

Now, $\langle \underline{7} \rangle$ is an element of 8 iff $\langle \underline{5} \rangle$ is an element of 9. So using f and the assumption that 2 is decidable, we can decide 10. A contradiction. Therefore, 2 is undecidable. (This also means that the complement of 2 is undecidable; the complement of any undecidable language is itself undecidable.)

Figure 1. Proof template that may be appropriate for question 4b on page 3.



NUI MAYNOOTH

Ollscoil na hÉireann Má Nuad



JANUARY 2009 EXAMINATION

CS605

The Mathematics and Theory of Computer Science

Dr. P. Sawyer, Dr. A. Winstanley, Mr. T. Naughton

Declaration

To be signed by the student and collected by an
invigilator at the beginning of the examination

1. I have searched through my copies of M. Sipser, *Introduction to the Theory of Computation*, first and second editions, (the Sipser book) and it does not contain any extra pages or annotations (except for annotations that correct minor typographical errors).
2. I understand that by failing to notify an invigilator of any annotations or extra pages in my copies of the Sipser book, I will receive a mark of zero in this examination. This does not affect any further disciplinary actions that the University may wish to take.
3. I understand also that directly copying large amounts of material from the Sipser books without substantially tailoring it to the question asked may result in a mark of zero.

Print name _____ Student number _____

Signed _____ Date _____