



NUI MAYNOOTH

Ollscoil na hÉireann Má Nuad

**OLLSCOIL NA hÉIREANN MÁ NUAD**  
**NATIONAL UNIVERSITY OF IRELAND MAYNOOTH**

**M.Sc. in Software Engineering Examination**

**SEMESTER 1**  
**2004-2005**

**MATHEMATICS AND THEORY OF COMPUTER SCIENCE**  
**PAPER CS605**

Dr. Ita Richardson, Prof. Ronan Reilly, Mr. Tom Naughton

Time allowed: 3 hours

Answer *three* questions

**All questions** carry equal marks

**Additional material allowed:**

**One copy of M. Sipser, *Introduction to the Theory of Computation* (PWS, Boston, 1997),  
containing no annotations or extra pages. Please see declaration page at back.**

1. (a) For each of the finite automata in Figure 1 on page 3 give a regular expression that generates the language that the finite automaton accepts. [6 marks]
- (b) How can one use a reduction to prove nonmembership of a class? [3 marks]
- (c) Prove that every nondeterministic finite automaton can be converted into an equivalent one that has a single accept state. Your proof should consist of an unambiguous sequence of steps that describes how to construct the equivalent machine. [6 marks]
- (d) The language  $\text{HALT}$  is defined  $\text{HALT} = \{\langle M, w \rangle : M \text{ is a Turing machine and } M \text{ halts on input } w\}$ . Prove that  $\text{HALT}$  is undecidable. You cannot use the fact that any other problem is known to be undecidable in your proof. [10 marks]
2. (a) Let  $\Sigma = \{1, +, =\}$  and let  $\text{ADD} = \{x=y+z : x, y, z \in \{1\}^*, |x| = |y| + |z|\}$ . [6 marks]  
Prove that  $\text{ADD}$  is not regular.
- (b) State three undecidable properties of C++ programs. [3 marks]
- (c) i. What would be the implications if a C++ algorithm that had exponential time-complexity was found to solve a  $\mathcal{NP}$ -complete problem? [3 marks]
- ii. What would be the implications if at least one  $\mathcal{NP}$  problem were found to have a polynomial Turing machine solution? [2 marks]
- (d) It is claimed that if a language is not countable then a Turing machine cannot accept it.
  - i. Prove this claim either true or false. [6 marks]
  - ii. What about the converse claim: that if a language is countable then a Turing machine can accept it? Prove this claim either true or false. [5 marks]
3. (a) i. Within a polynomial approximation, what can the computational complexity of a Turing machine solution to a problem tell us about the complexity of a C++ solution to that problem? [3 marks]
- ii. What thesis supports your answer? [1 marks]
- iii. State this thesis. [1 marks]
- (b) Given a countable set  $A$  containing all of the subsets of  $X$  (i.e. for all  $a \subseteq X, a \in A$ ) prove that  $X$  must be finite. [5 marks]
- (c) A *useless state* is a state in a machine that is never entered on any input word. [15 marks]  
Consider the problem of testing whether a machine has any useless states. For each of the following two cases, formulate this problem as a language and prove that it is decidable or prove that it is undecidable. (If an undecidability proof is appropriate, you can choose to assign a name, mathematical construct, or piece of pseudocode to each of the numbered blanks in the proof template in Figure 2 on page 3. Where blanks have the same number, this denotes their contents will be the same. In such a case, you can assume that  $A_{\text{TM}} = \{\langle M, w \rangle : M \text{ is a Turing machine and } M \text{ accepts } w\}$  is undecidable.)
  - i. Let the machines in question be Turing machines.
  - ii. Let the machines in question be finite automata.

4. (a) The playing of certain board games poses intractable problems for machines. [5 marks]  
One such game is called GOSLOWLY. It is not necessary to understand the rules of GOSLOWLY, but you are told that the language corresponding to the problem of deciding which player is the winner from the state of the board is  $\mathcal{NP}$ -hard. However, it is claimed that this language is not  $\mathcal{NP}$ -complete. Is this even possible? Explain.
- (b) Let  $L = \{w : w \in \{a, b\}^*, w \text{ contains an equal number of } as \text{ and } bs\}$ .
- Construct a Turing machine (include the full table of behaviour) to decide  $L$ . [7 marks]
  - Calculate the exact (not asymptotic approximation) time-complexity of your algorithm in terms of number of transitions (tape head movements). [3 marks]
  - Construct a pushdown automaton (give the full state diagram) to accept  $L$ . [5 marks]
  - Give a context-free grammar that generates  $L$ . [3 marks]
  - Can  $L$  be accepted by a deterministic pushdown automaton? Justify your answer. [2 marks]

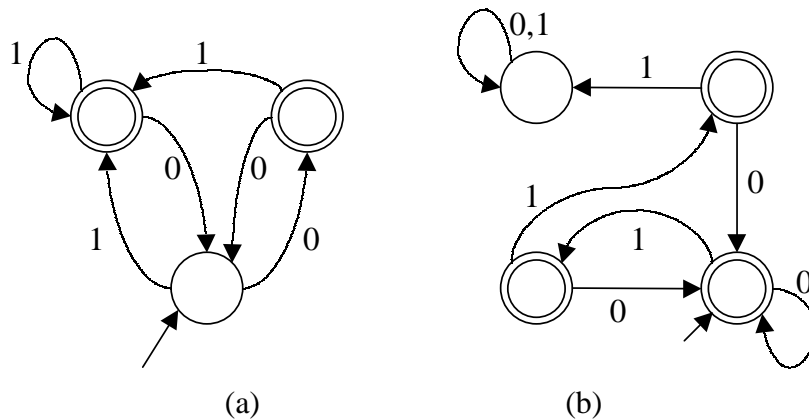


Figure 1: Finite automata for question 1a.

**Proof.** We will use a mapping reduction to prove the reduction 1. Assume that 2 is decidable. The function  $f$  that maps instances of 3 to instances of 4 is performed by TM  $F$  given by the following pseudocode.

$F =$  “On input  $\langle \_5 \rangle$  :

1. Construct the following  $M'$  given by the following pseudocode.

$M' =$  “6”

2. Output  $\langle \_7 \rangle$ ”

Now,  $\langle \_7 \rangle$  is an element of 8 iff  $\langle \_5 \rangle$  is an element of 9. So using  $f$  and the assumption that 2 is decidable, we can decide 10. A contradiction. Therefore, 2 is undecidable. (This also means that the complement of 2 is undecidable; the complement of any undecidable language is itself undecidable.)

Figure 2: Proof template that may be appropriate for question 3c.

**OLLSCOIL NA hÉIREANN MÁ NUAD**  
**NATIONAL UNIVERSITY OF IRELAND MAYNOOTH**

**M.Sc. in Software Engineering Examination**

**SEMESTER 1**  
**2004-2005**

**MATHEMATICS AND THEORY OF COMPUTER SCIENCE**  
**PAPER CS605**

---

Dr. Ita Richardson, Prof. Ronan Reilly, Mr. Tom Naughton

---

**Declaration**

**To be signed by the student and collected by an  
invigilator at the beginning of the examination**

1. I have searched through my copy of M. Sipser, *Introduction to the Theory of Computation*, PWS, Boston 1997 (the Sipser book) and it does not contain any extra pages or annotations (except for annotations that correct minor typographical errors).
2. I understand that by failing to notify an invigilator of any annotations or extra pages in my copy of the Sipser book, I will receive a mark of zero in this examination. This does not affect any further disciplinary actions that the University may wish to take.
3. I understand also that directly copying large amounts of material from the Sipser book without substantially tailoring it to the question asked may result in a mark of zero.

Print name \_\_\_\_\_ Student number \_\_\_\_\_

Signed \_\_\_\_\_ Date \_\_\_\_\_