OLLSCOIL NA hÉIREANN, MÁ NUAD

NATIONAL UNIVERSITY OF IRELAND, MAYNOOTH

M.Sc. SOFTWARE ENGINEERING EXAMINATION

SAMPLE 2002/2003

PAPER CS605

MATHEMATICS AND THEORY OF COMPUTER SCIENCE

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Answer three (3) questions from four (4). Time Allowed: 3 hours.

Note, in this sample paper, some questions do not total to 25 marks. For the exam paper, each question will be worth exactly 25 marks. Furthermore, there will be exactly four questions on the exam paper.

1. (a) Expand the languages defined by the following expressions. Note, e denotes the empty word, $\circ$ denotes concatenation. [6 marks]
   i. $\emptyset$
   ii. $\emptyset \circ \{\emptyset\}$
   iii. $\emptyset \circ \{a, b\}$
   iv. $\{\emptyset\} \circ \{a, b\}$
   v. $\{e\}^*$
   vi. $\{e\} \circ \{a, b\}^*$

2. Let the alphanumeric set of symbols be defined as $A = (\{a, b, ...z\} \bigcup \{0, 1, ...9\})^*$. Let f be an arbitrary computer program that takes as arguments four alphanumeric strings and returns an integer. Let $X = (x_1, x_2, x_3, x_4)$ be an arbitrary input to f, where each $x_i \in A$. Let α be an arbitrary output of f, where $\alpha \in \mathbb{Z}$. Consider the following problem. Given an arbitrary f and an arbitrary α, is there an input X to f that generates α? [25 marks]
   (a) Define a language that captures the essence of this decision problem.
   (b) Is this language recursively-enumerable? Prove your answer.
   (c) Is this language recursive? Prove your answer.

3. Refer to the problem in Question 2. Let us attempt to make the problem easier by modifying f so that this program is guaranteed to halt after some fixed number of timesteps. [25 marks]
   (a) Define a language that captures the essence of this new modified decision problem.
   (b) Is this language recursively-enumerable? Prove your answer.

(c) Is this language recursive? Prove your answer.

4. For each of the following languages, prove that it is regular by providing a (non-deterministic or deterministic) finite automaton to accept it, or prove that it is not regular through the use of the pumping lemma. [35 marks]

   (a) $\{w : w \in \{a, b\}^*, w$ contains the substring $aab\}$

   (b) $\{uxv : u \in (K \cup L), K = \{w : w \in \{a, b\}^*, w$ contains at most two as$\}, L = \{aab, bb\}, v \in \{a, b\}^*\}$

   (c) $\{ww : w \in \{a, b\}^*\}$

   (d) $\{0^m 1^n : m, n \geq 0, m \neq n\}$

   (e) $\{w : w \in \{0, 1\}^*, w$ contains a different number of 0s as 1s$\}$

   (f) $\{uv : u, v \in \{a, b\}^*, u$ is longer than $v\}$

   (g) $\{a^m b^n : m > n \geq 0\}$

   (h) $\{a^n b^n : 0 \leq n \leq 2^2\}$

5. For each language in Question 4, prove that it is context-free by providing a push-down automaton to accept it or a context-free grammar to generate it, or give a concise argument as to why it is not context-free. [25 marks]

6. (a) You have been shown a proof that the union or concatenation of two regular languages is a regular language. You have been shown a proof that the Kleene star of a regular language is a regular language. But what about the complement of a language? The complement of a language is defined as follows. Given a language L over an alphabet $\{a, b\}$ the complement of L is the set of all words in $\{a, b\}^*$ that are not in L, written more formally as $\overline{L} = \{w : w \in \{a, b\}^*, w \notin L\}$. Prove that the complement of a regular language is a regular language. [5 marks]

   (b) Language $L = \{w : w \in \{a, b\}^*, w$ contains the substring aab and contains no more that three bs in total$\}$ is a regular language. The pumping lemma says that every word in L that is greater than the pumping length p, can be written as the concatenation of three strings xyz such that pumping y results in another word in L. There are two restrictions on how the word can be written in xyz format: [5 marks]

   - y has to contain at least one symbol,
   - $|xy| \leq p$.

   Illustrate that you know what the pumping lemma means by picking any three words in L that have length at least $p = 5$, and rewrite them in xyz form such that the y part can be pumped.

7. (a) Prove that the language $L = \{w : w \in \{a, b\}^*, w$ contains twice as many as as bs$\}$ cannot be pumped, by finding at least one word in L, with length greater than p, that cannot be written in the xyz form. (This proves that L is nonregular.) [3 marks]

(b) Prove that the language $L = \{w : w \in \{a, b\}^*, w$ contains the same number of occurrences of ab as $ba\}$ is regular, by supplying a nondeterministic finite automaton or regular expression to describe it, or prove that it is nonregular using the pumping lemma. [5 marks]

8. ... *various other questions on recursive languages, recursively enumerable languages, undecidable problems, $\mathcal{NP}$-completeness, etc.* [25 marks]